Symmetries, Colorings, and Polyanumeration

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Master's Thesis under Supervision of Dr. Hossein Shahmohamad

1 Introduction

Consider a cube floating in space: how many unique ways can we color that cube with two colors uniquely? If the cube were fixed, the problem would simplify to two color choices for each face of the cube, with six faces. In other words, $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$. A floating cube becomes a more difficult problem due to the cube's geometric symmetries. We can develop a formula based on these symmetries to count the distinct colorings of a floating figure. With only a little more work, we can also obtain a generating function for the *pattern inventory* of the figure in question.

Example 1.1: Pattern Inventory

The pattern inventory of the black-white colorings of a cube with all geometric symmetries allowed is;

$$b^8 + b^7w + 3b^6w^2 + 3b^5w^3 + 7b^4w^4 + 3b^3w^5 + 3b^2w^6 + bw^7 + w^8.$$

To find the number of distinct colorings of a cube in three space, we must find a way to partition it into subsets which are composed of equivalent elements. First, however, it is necessary to understand the general concept of how two elements (a and b) may be equivalent (written $a \sim b$). We consider the fundamental properties of an equivalence relation to be:

(i) Transitivity: $a \sim b$, $b \sim c \Rightarrow a \sim c$
(ii) Reflexivity: $a \sim a$
(iii) Symmetry: $a \sim b \Rightarrow b \sim a$

Any other properties of equivalence can be derived from these three. Any binary relation with these properties is called an *equivalence relation*. Any equivalence relation defines a partition into subsets of mutually equivalent elements called *equivalence classes*.

Example 1.2: Equivalence Relations

For a set of numbers, an equivalence relation could be defined as differing by an even number. In this way, the even numbers and the odd numbers form separate equivalence classes.

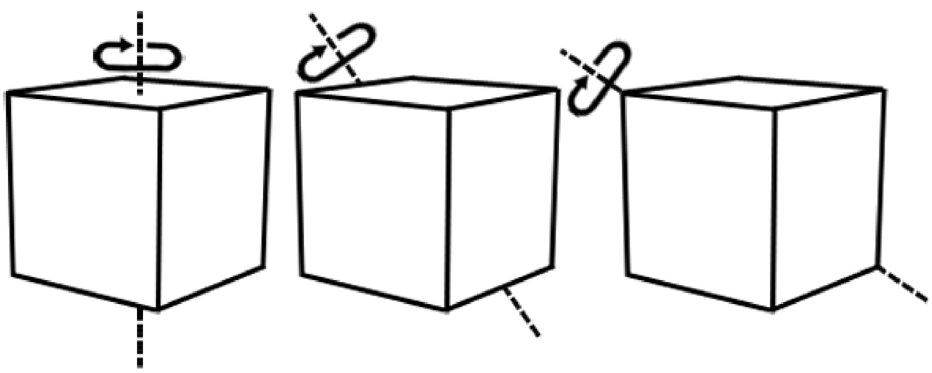
To take a closer look at geometric symmetries, we observe what sort of motions map a cube onto itself. These motions can be broken up into three types, all rotations around an axis in the cube. The first type of rotation is the rotation around the axis built when we join the centers of two opposite faces to each other. Three axes can be built this way, and around these axes, we can rotate the cube 90^0, 180^0, and 270^0. The second type of rotation we use is one around the axis created when we join two opposite vertices of the cube. Four axes can be built this way. Around these axes we can rotate the cube 120^0 and 240^0. The third type of rotation is one around the axis created when we join midpoints of two opposite edges of the cube together. With this method, six axes can be built, and around each of these axes we can rotate the cube 180^0.

If we label the faces of a square as a, b, c, d, e and f, we can categorize the way symmetries permute the faces of a cube. Thus, a 90^0 rotation of the first type can be described as the permutation: $a \to b$, $b \to c$, $c \to d$, $d \to a$, $e \to e$, $f \to f$. A 120^0 rotation of the second type could be described as: $a \to b$, $b \to e$, $e \to a$, $c \to f$, $f \to d$, $d \to c$, or $a \to b \to e \to a$, $c \to f \to d \to c$. A permutation in the form $x_1 \to x_2 \to ... \to x_n \to x_1$ is called a *cycle*. Thus, the permutation created by the 90^0 rotation of the first type, described as $a \to b \to c \to d \to a$, is a cycle of length four. The form cycles are normally written in is $(x_1 x_2 ... x_n x_1)$, so the aforementioned cycle is written $(abcd)$.

Remark: Any permutation can be expressed as a product of disjoint cycles

For example, the permutation described above for the 120^0 rotation of the second type could be described in the form $(abe)(cfd)$. We can list the symmetries of a figure and label each successive one with an appropriate π_i. What we really are interested in is the way each symmetry (each π_i) takes one coloring into another, making them equivalent. Figure 1.1 shows the rotations of the cube.

Figure 1.1: Symmetries of a cube
Type 1, Type 3, Type 2



We define the equivalence relation of colorings by claiming C and C' are equivalent $(C \sim C')$ if there exists a symmetry π_i such that $\pi_i(C) = C'$.

To show that this is an equivalence relation, we look at the properties of the set of symmetries (called G) that make the relation $C \sim C'$ an equivalence relation. These properties are:

(i) Closure: If $\pi_i, \pi_j \epsilon G$, then $\pi_i \pi_j \epsilon G$ (Satisfies transitivity)
(ii) Identity: G contains an identity motion π_1 such that $\pi_i \pi_1 = \pi_1 \pi_i = \pi_i$ (Satisfies reflexivity)
(iii) Inverses: For each $\pi_i \epsilon G$, there exists an inverse in G, denoted π_i^{-1} such that $\pi_i^{-1}\pi_i = \pi_i \pi_i^{-1} = \pi_1$ (Satisfies Symmetry)

Any collection G of objects that has a binary operation that satisfies the above properties, as well as the associative property $((\pi_i \pi_j)\pi_k = \pi_i(\pi_j \pi_k))$ we consider to be a *group*.

Theorem 1.1: Let G be a group of permutations of the set S and T be any collection of colorings of S. Then G induces a partition of T into equivalence classes with the relation $C \sim C'$ if and only if some $\pi \epsilon G$ takes C to C'.

Note that S could be any set of objects and T could be any collection. In our current example we consider S to be the set of the faces of a cube, and T to be the two-coloring of the faces of a cube.

Lemma 1.2: For any two permutations π_i, π_j in a group G there exists a unique permutation $\pi_k = \pi_i^{-1}\pi_j$ in G such that $\pi_i \pi_k = \pi_j$.

Proof: First we show that $\pi_i \pi_k = \pi_j$. Since $\pi_k = \pi_i^{-1}\pi_j$, then by associativity $\pi_i \pi_k = \pi_i(\pi_i^{-1}\pi_j = (\pi_i \pi_i^{-1})\pi_j = \pi_1 \pi_j = \pi_j$ as claimed. Next suppose there also exists a permutation π_k' such that $\pi_i \pi_k' = \pi_j$. Then $\pi_i \pi_k = \pi_i \pi_k'$. Multiplying the equation by π_i^{-1}, we have $\pi_i^{-1}(\pi_i \pi_k) = \pi_i^{-1}(\pi_i \pi_k') \Rightarrow (\pi_i^{-1}\pi_i)\pi_k = (\pi_i^{-1}\pi_i)\pi_k' \Rightarrow \pi_1 \pi_k = \pi_1 \pi_k' \Rightarrow \pi_k = \pi_k'$

Now, we seek the number (N) of equivalence classes of T (colorings of our cube) induced by a group G of symmetries of our cube. Suppose there is a group of s symmetries acting on c colorings in T. Let E_C be the equivalence class consisting of C and all colorings C' equivalent to C. In other words, all C' such that for some $\pi \epsilon G$, $\pi(C) = C'$. Note that if all s of the πs takes C to a different coloring (ie: $\pi(C)$), then E_C would have s colorings. Also note that $\pi_1(C) = C$, so C is included in $\pi(C)$. If every equivalence class is like this with s colorings, then $sN = c$ (ie: (total number of symmetries) x (number of equivalence classes) = (the number of colorings)). Solving for N, we have $N = \frac{c}{s}$.

Example 1.3: Number of Equivalence Classes

Consider the $c = n!$ oriented seatings of n people around a round table. There are $s = n$ cyclic rotations of the seatings, and each equivalence class consists of n seatings. Thus the number of equivalence classes (cyclic nonequivalent seatings) is $N = \frac{n!}{n} = (n-1)!$.

On the other hand, suppose we have a small round table with three positions for chairs. With only and white and black chairs are available. There are $2^3 = 8$ ways to place a white or black chair in each position. There are three cyclic rotations of the table possible, 0^0, 120^0, and 240^0. we have $c = 8$ colorings and $s = 3$ symmetries, but the number of equivalence classes cannot be $N = \frac{8}{3}$, it is a fraction. We need to correct our formula by adding the multiplicities of an arrangement. When one or more symmetries map the arrangement to itself instead of other arrangements. When all the multiplicities are counted, every equivalence class will have s members. Since two symmetries (along with the 0^0 symmetry) leave the all black chair arrangement fixed (along with the all white chair arrangement), this arrangement has the multiplicity of three. We can correct our formula by adding together the multiplicities of the different colorings, to get $N = \frac{3+1+1+1+1+1+1+3}{3} = 4$. Table 1.2 gives us the colorings of a cube if it were fixed in three dimensions, and Table 1.1 shows the symmetries that return a cube to itself in three dimensions.

Table 1.1: Symmetries of a Cube with Cycles

Symmetry	Cycle	Symmetry	Cycle
π_1	$(a)(b)(c)(d)(e)(f)$	π_{13}	$(afd)(bce)$
π_2	$(abcd)(e)(f)$	π_{14}	$(adf)(bec)$
π_3	$(ac)(bd)(e)(f)$	π_{15}	$(ade)(bfc)$
π_4	$(adcb)(e)(f)$	π_{16}	$(aed)(bcf)$
π_5	$(aecf)(b)(d)$	π_{17}	$(afb)(edc)$
π_6	$(ac)(b)(d)(ef)$	π_{18}	$(abf)(ecd)$
π_7	$(afce)(b)(d)$	π_{19}	$(ad)(bc)(ef)$
π_8	$(a)(bfde)(c)$	π_{20}	$(ab)(cd)(ef)$
π_9	$(a)(bd)(c)(ef)$	π_{21}	$(ae)(bd)(cf)$
π_{10}	$(a)(bedf)(c)$	π_{22}	$(af)(bd)(ce)$
π_{11}	$(abe)(cfd)$	π_{23}	$(ac)(be)(df)$
π_{12}	$(aeb)(cdf)$	π_{24}	$(ac)(bf)(de)$

Table 1.2: Colorings of a fixed cube

faces are a, b, c, d, e, and f

Coloring	White Faces	Black Faces	Coloring	White Faces	Black Faces
C_1	a, b, c, d, e, f	none	C_{33}	b, c, d, e, f	a
C_2	a, b, c, d, e	f	C_{34}	b, c, d, e	a, f
C_3	a, b, c, d, f	e	C_{35}	b, c, d, f	a, e
C_4	a, b, c, d	e, f	C_{36}	b, c, d	a, e, f
C_5	a, b, c, e, f	d	C_{37}	b, c, e, f	a, d
C_6	a, b, c, e	d, f	C_{38}	b, c, e	a, d, f
C_7	a, b, c, f	d, e	C_{39}	b, c, f	a, d, e
C_8	a, b, c	d, e, f	C_{40}	b, c	a, d, e, f
C_9	a, b, d, e, f	c	C_{41}	b, d, e, f	a, c
C_{10}	a, b, d, e	c, f	C_{42}	b, d, e	a, c, f
C_{11}	a, b, d, f	c, e	C_{43}	b, d, f	a, c, e
C_{12}	a, b, d	c, e, f	C_{44}	b, d	a, c, e, f
C_{13}	a, b, e, f	c, d	C_{45}	b, e, f	a, c, d
C_{14}	a, b, e	c, d, f	C_{46}	b, e	a, c, d, f
C_{15}	a, b, f	c, d, e	C_{47}	b, f	a, c, d, e
C_{16}	a, b	c, d, e, f	C_{48}	b	a, c, d, e, f
C_{17}	a, c, d, e, f	b	C_{49}	c, d, e, f	a, b
C_{18}	a, c, d, e	b, f	C_{50}	c, d, e	a, b, f
C_{19}	a, c, d, f	b, e	C_{51}	c, d, f	a, b, e
C_{20}	a, c, d	b, e, f	C_{52}	c, d	a, b, e, f
C_{21}	a, c, e, f	b, d	C_{53}	c, e, f	a, b, d
C_{22}	a, c, e	b, d, f	C_{54}	c, e	a, b, d, f
C_{23}	a, c, f	b, d, e	C_{55}	c, f	a, b, d, e
C_{24}	a, c	b, d, e, f	C_{56}	c	a, b, d, e, f
C_{25}	a, d, e, f	b, c	C_{57}	d, e, f	a, b, c
C_{26}	a, d, e	b, c, f	C_{58}	d, e	a, b, c, f
C_{27}	a, d, f	b, c, e	C_{59}	d, f	a, b, c, e
C_{28}	a, d	b, c, e, f	C_{60}	d	a, b, c, e, f
C_{29}	a, e, f	b, c, d	C_{61}	e, f	a, b, c, d
C_{30}	a, e	b, c, d, f	C_{62}	e	a, b, c, d, f
C_{31}	a, f	b, c, d, e	C_{63}	f	a, b, c, d, e
C_{32}	a	b, c, d, e, f	C_{64}	none	a, b, c, d, e, f

For our example (the two color face colorings of a cube), the multiplicity correction is more complicated. Note that the size of an equivalence class can be one, two, three, four, six, eight, or twelve. It, however, cannot be twenty four, the the number of symmetries of the cube. We run into two major problems here. The first is that some of the πs besides the identity symmetry π_1 may leave a coloring fixed (ie: $\pi(C_i) = C_i$). The other is that there may be several πs taking C_i to another coloring C_k.

Let C_4, and C_{41} be the colorings described in Table 1.2. Also let π_2, π_5, and π_7 be the symmetries described in Table 1.1. Note that $\pi_2(C_4) = C_4$. This is an example of our first problem. To show our second problem, notice $\pi_5(C_4) = \pi_7(C_4) = C_{41}$. Also, note that $\pi_7 = \pi_5\pi_6$. We seek a conjecture to straighten out these problems.

Conjecture 1.3: Let E be an equivalence class with s elements. Now, let $\phi(x)$ denote the number of πs that leave the coloring x fixed. Then, $\sum_{x \epsilon E} \phi(x) = s$.

Proof: Let $x_1, x_2, ..., x_m$ be the colorings in equivalence class E and let $\pi_1, \pi_2, ..., \pi_s$ be the group of symmetries. These πs can be divided into m groups, $R_1, R_2, ..., R_m$, where R_i is the set of πs that map x_1 to x_i. To show that the number of πs mapping x_1 to x_i equals $\phi(x_i)$, the number of πs that leave x_i fixed, consider any $\pi*$ that leaves x_i fixed and one map, π' that takes x_1 to x_i. Then $\pi'\pi*$ also takes x_1 to x_i. so, for each $\pi*$, $\pi'\pi * \epsilon R_i$. Thus there is a 1-1 correspondence between the $\pi*$s and the elements of R_i. Thus $\sum_{x \epsilon E} \phi(x)$ sums the number of πs in R_1, in R_2, ... , and in R_m. This sum is just the total number of symmetries, thus s.

If we sum over all equivalence classes, we obtain an important theorem, first proved by Burnside.

Theorem 1.4: Let G be a group of permutations of the set S. Let T be any collection of colorings of S that is closed under G. Then the number N of equivalence classes is:

$$N = \frac{1}{|G|} \sum_{x \epsilon T} \phi(x)$$

or:

$$N = \frac{1}{|G|} \sum_{\pi \epsilon G} \psi(\pi)$$

where $|G|$ is the number of permutations and $\psi(\pi)$ is the number of colorings in T left fixed by π.

Note that, by *closed under* G we mean that for all $\pi \epsilon G$ and $x \epsilon T, \pi(x) \epsilon T$. Both formulas in the theorem above count all instances of some coloring being left fixed by some π, the first sums over the different colorings, while the second sums over different πs.

To informally summarize the second formula above, the total number c of all colorings of our cube is equal to $\psi(\pi_1)$, since the identity symmetry π_1 leaves all colorings fixed. If each of the 24 symmetries mapped a coloring C into 24 different colorings, then we would have 24 colorings in each equivalence class and hence a total of $\frac{c}{24} = \frac{\psi(\pi_1)}{|G|}$ equivalence classes. However, the colorings in the collection $\pi(C) : \pi \epsilon G$ forming an equivalence class of colorings of a cube are never distinct for a cube. The terms $\phi(\pi_i)$ in the second sum above add in the *multiplicities* of repeated colorings so that each equivalence class has 24 colorings in the second sum.

Unfortunately, without further work, it is difficult to apply Theorem 1.4 to counting different colorings of an unoriented figure. What we need is a way to determine $\phi(\pi)$ from the structure of π. Let us apply Theorem 1.4 to our cube. When we first counted the 2-colorings of our cube, we did so by inspection. This time as we count $\psi(\pi_i)$ for each π_i, we look for a pattern that would allow us to predict mathematically which colorings will be left fixed by π_i. See Tables 1.1 and 1.2 for colorings of a cube and the group of symmetries respectively. Now, let us construct a table of the π_is and the colorings they leave fixed.

Table 1.3: Symmetries and the colorings they leave fixed

π_i	**Colorings Left Fixed by π_i**	**Cycle Structure Representation**	π_i	**Colorings Left Fixed by π_i**	**Cycle Structure Representation**
π_1	64	x_1^6	π_{13}	4	x_3^2
π_2	8	$x_4x_1^2$	π_{14}	4	x_3^2
π_3	16	$x_2^2x_1^2$	π_{15}	4	x_3^2
π_4	8	$x_4x_1^2$	π_{16}	4	x_3^2
π_5	8	$x_4x_1^2$	π_{17}	4	x_3^2
π_6	16	$x_2^2x_1^2$	π_{18}	4	x_3^2
π_7	8	$x_4x_1^2$	π_{19}	8	x_2^3
π_8	8	$x_4x_1^2$	π_{20}	8	x_2^3
π_9	16	$x_2^2x_1^1$	π_{21}	8	x_2^3
π_{10}	8	$x_4x_1^2$	π_{22}	8	x_2^3
π_{11}	4	x_3^2	π_{23}	8	x_2^3
π_{12}	4	x_3^2	π_{24}	8	x_2^3

The identity rotation (π_1) leaves each face fixed and hence leaves all colorings fixed, so $\psi(\pi_1) = 64$. More interestingly, π_2 cyclically permutes faces a, b, c and d. Note that for a color to be left fixed by π_2 means that the face has the same color after the rotation as it did before. Since π_2 takes a to b, then a coloring left fixed by π_2 must have the same color on faces a and b. Notice that this causes faces a, b, c and d to have the same color for π_2 to leave a particular coloring fixed. Since π_2 leaves faces e and f fixed, the number of colorings π_2 leaves fixed is equal to the two color choices for the cycle $(abcd)$, times the two color choices for e, times the two color choices for f. So, $\psi(\pi_2) = 2^3 = 8$. In general we can say that a coloring C will be left fixed by π if and only if for each face v, the color of C at v is the same as the color at $\pi(v)$ so that the symmetry leaves the color at $\pi(v)$ unchanged.

From here we can see that, as π_3 leaves e and f fixed and cyclically permutes a to c and b to d, then $\psi(\pi_3) = 2^4 = 16$. Note that π_4 is a 270^0 rotation, which will be similar to π_2. We note that $\psi(\pi_2) = \psi(\pi_4) = 8$. A pattern for the first type of rotations should be coming clear. Let us take a look at the rotations of the second type. The rotation π_{11} cyclically permutes a to b to e and c to f to d. So, $\psi(\pi_{11}) = 2 \times 2 = 4$. This same pattern will follow all rotations of the second type. As an example of rotations of the third type, notice π_{19} cyclically permutes a to d, b to c, and e to f. Thus, $\psi(\pi_{19}) = 2 \times 2 \times 2 = 8$. All rotations of the third type are similar to π_{19}.

For use in the future, we should classify cycles by their length. It will be useful to create a product of these cycles, using the form of the cycle notation we developed earlier, such that x_1 represents the cycles of length one in π_i, x_2 represents the cycles of length two in π_i, and so on. We call this product the *cycle structure representation* for the symmetry. For example, as we can see in Table 1.3, the cycle structure representation of π_2 is $x_4 x_1^2$ while the cycle structure representation of π_{20} is x_2^3. Thus we can "predict" that π_2 leaves $2x2^2 = 8$ colorings fixed, and that π_{20} leaves $2^3 = 8$ colorings fixed.

Conjecture 1.5: For any π_i, the number of colorings left fixed will be given by setting each x_j equal to the number of colors we wish to color the object with (in our case, 2) in the cycle structure representation of π_j, that is,

$$\psi(\pi) = 2^{\textit{number of cycles in } \pi}$$

So, to obtain the number of different 2-colorings of the cube with Theorem 1.4, we sum the second column of Table 1.3 and divide by 24.

$$\frac{1}{|G|} \sum_{\pi \epsilon G} \psi(\pi) = \frac{1}{24}(240) = 10$$

A simpler way to obtain this result is to first algebraically sum the cycle structure representations from each symmetry, collecting like terms together, and then divide by 24. From the third and sixth columns of Table 1.3, we obtain $\frac{1}{24}(x_1^6 + 6x_4x_1^2 + 3x_2^2x_1^2 + 8x_3^2 + 6x_2^3)$. This expression is called a *cycle index* $P_G(x_1, x_2, ..., x_k)$ for a group of G symmetries. By setting each $x_i = 2$ in this cycle index (ie: $P_G(2, 2, ..., 2)$) we get the same answer. More generally, we have the following Theorem.

Theorem 1.6: Let S be a nonempty set of elements and G be a group of symmetries of S that acts to induce an equivalence relation on the set of m-colorings of S. Then the number of nonequivalent m-colorings of S is given by $P_G(m, m, ..., m)$.

Now recall that we previously defined a pattern inventory as a generating function that tells how many colorings of an unoriented figure there are using different possible collections of colors. For the black-white colorings of our cube, the pattern inventory is $b^6+b^5w+2b^4w^2+2b^3w^3+2b^2w^4+bw^5+w^6$. The term $2b^4w^2$ tells us that there are two nonequivalent colorings with four black faces and two white faces. Note that the coefficients in the pattern inventory can be considered to be the results of several counting problems related to Theorem 1.4.

In the case of the cube, we divide the set of all colorings into sets based on the numbers of black and white faces:

$T_0 = (C_1)$
$T_1 = (C_2, C_3, C_5, C_9, C_{17}, C_{33})$
$T_2 = (C_4, C_6, C_7, C_{10}, C_{11}, C_{13}, C_{18}, C_{19}, C_{21}, C_{25}, C_{34}, C_{35}, C_{37}, C_{41}, C_{49})$
$T_3 = (C_8, C_{12}, C_{14}, C_{15}, C_{20}, C_{22}, C_{23}, C_{26}, C_{27}, C_{29}, C_{36}, C_{38}, C_{39}, C_{42}, C_{43}, C_{45}, C_{50}, C_{51}, C_{53}, C_{57})$
$T_4 = (C_{16}, C_{24}, C_{28}, C_{30}, C_{31}, C_{40}, C_{44}, C_{46}, C_{47}, C_{52}, C_{54}, C_{55}, C_{58}, C_{59}, C_{61})$
$T_5 = (C_{32}, C_{48}, C_{56}, C_{60}, C_{62}, C_{63})$
$T_6 = (C_{64})$

The coefficient of b^4w^2 can be obtained from the second equation in Theorem 1.4 if we let the group G of symmetries of the cube act on just the set T_4. Note that T_4 is closed under G since any symmetry acting on a coloring with four black faces and two white faces returns another such coloring. The same is true for the other T_ks, thus the coefficient of b^kw^{4-k} in the pattern inventory is the result of the second equation of Theorem 1.4 when T_k is the set on which G acts.

Now, we try to solve the seven problems simultaneously. In Table 1.4, we write the polynomials whose coefficients represent the number of 2-colorings in each T_k left fixed by π_i for each motion π. Then we total up the b^6 term in each row (the number of 2-colorings with six black faces) and divide by twenty-four to get the coefficient of b^6 in the patten inventory, then repeat the process for each coefficient.

Since the action π_1 leaves all C_s fixed, the first row's coefficients are 1,6,15,20,25,15,6,1. We write $b^6 + 6b^5w + 15b^4w^2 + 20b^3w^3 + 15b^2w^4 + 6bw^5 + w^6$; this is an inventory of fixed colorings. Note that for π_1, the inventory of fixed colorings is an inventory of all colorings, also note that this inventory is simply $(b+w)^6 = (b+w)(b+w)(b+w)(b+w)(b+w)(b+w)$. For π_2 this inventory is $b^6 + 2b^5w + b^4w^2 + b^2w^4 + 2bw^5 + w^6$, which factors into $(b^4 + w^4)(b + w)^2$. Hopefully, a pattern emerges as it did before.

Table 1.4: Inventory of Colorings Left Fixed by each π_i

π_i**s**	**Polynomial**	**Expanded polynomial**
π_1	$(b+w)^6$	$b^6+6b^5w+15b^4w^2+20b^3w^3+15b^2w^4+6bw^5+w^6$
$\pi_2,\pi_4,\pi_5,\pi_7,\pi_8,\pi_{10}$	$(b^4+w^4)(b+w)^2$	$b^6+2b^5w+b^4w^2+b^2w^4+2bw^5+w^6$
π_3,π_6,π_9	$(b^2+w^2)^2(b+w)^2$	$b^6+2b^5w+3b^4w^2+4b^3w^3+3b^2w^4+2bw^5+w^6$
$\pi_{11},\pi_{13},\pi_{15},\pi_{17}$	$(b^3+w^3)^2$	$b^6+2b^3w^3+w^6$
$\pi_{12},\pi_{14},\pi_{16},\pi_{18}$	$(b^3+w^3)^2$	$b^6+2b^3w^3+w^6$
$\pi_{19},\pi_{20},\pi_{21},\pi_{22},\pi_{23},\pi_{24}$	$(b^2+w^2)^3$	$b^6+3b^4w^2+3b^2w^4+w^6$

The key to the pattern this time is similar, the fact that in each π all faces in a cycle of π must have the same color. Since π_2 has the cycle $(abcd)$, all four of those faces must have the same color for π_2 to leave the coloring fixed. Including the two other one-cycles, π_2 has the pattern inventory $(b^4+w^4)(b+w)^2$. In general, the inventory of fixed colorings for π_i will be a product of factors (b^j+w^j), one factor for each cycle j-cycle of the π_i. So, what we are really looking for is the number of cycles in π_i of each size. If we compare this to the information we decoded from the cycle structure notation. Compare the second column in Table 1.4 with the third and sixth columns in Table 1.3, and notice the similarities between the two. Setting each $x_j=(b^j+w^j)$ in the representation gives us the inventory of fixed colorings for π_i. To simplify the formula and save computation, we can proceed as before and first add together the cycle structure representations and dividing by the number of motions (in our case 24), and then setting each $x_j=(b^j+w^j)$ and doing the polynomial algebra all at once. The first step in this approach gives us the cycle index $P_G(x_1,x_2,...,x_k)$. Thus by setting $x_j=(b^j+w^j)$ in P_G, we obtain the pattern inventory.

Note that if three colors are used (say black, white and red), each cycle of size j would have an inventory of $(b^j+w^j+r^j)$ in a fixed coloring. In this case, we would set $x_j=(b^j+w^j+r^j)$ in P_G. This applies for any number of colors and any figure. With more generality, we have the following theorem.

Theorem 1.7: Let S be a set of elements and G be a group of permutations of S that acts to induce an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using two colors is given by the generating function $P_G((b+w),(b^2+w^2),...,(b^k+w^k)$. The inventory using colors $c_1,c_2,...,c_m$ is $P_G(\sum_{j=1}^m c_j, \sum_{j=1}^m c_j^2, ..., \sum_{j=1}^m c_j^k)$.

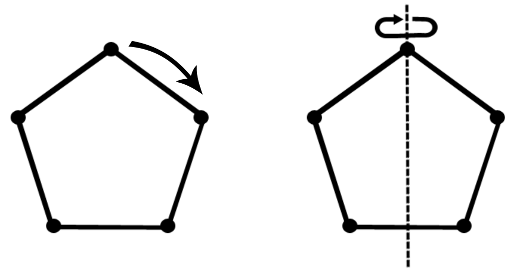
We can use this formula to count the total number of nonequivalent 2-colorings of our cube as well, as this number is just the sum of the coefficients of the pattern inventory. to sum these coefficients, we set the indeterminants b and w equal to one, or equivalently set each $x_j=2$ in P_G. If m colors were allowed, we would set $x_j=m$ in P_G, obtaining our earlier formula for the total number of colorings.

2 2-Dimensional Applications

Application 2.1: A Floating Pentagon

Consider the 2-colorings of the vertices of a floating pentagon. If the pentagon were fixed, we would simply have $2^5 = 32$ colorings of the oriented figure, but since the pentagon is unoriented, we need to look at the geometric symmetries of the object. Since the object is floating in space, it can be flipped along axes as well as rotated. We can rotate a regular pentagon five times; $72^0, 144^0, 216^0, 288^0$, and the identity rotation 0^0. As the object is floating, we also have five "flips," one around a line drawn between each vertex and the midpoint on the opposite edge. Figure 2.1 shows these rotations.

Figure 2.1: Symmetries of a Pentagon



Assume we can label the vertices a, b, c, d, and e. Then each of the rotations, which are not the identity rotation, have the cycle $(abcde)$. The "flips" on the other hand each have one vertex which stays fixed, and two 2-cycles. For example, if we flip around the line from a to the midpoint of cd, then the cycle structure is $(a)(be)(cd)$. From here, we can get the cycle structure representation in Table 2.1.

Table 2.1: Cycle Structure Representation of a Pentagon

Symmetry	Description	Cycles	Cycle Representation
π_1	Identity	$(a)(b)(c)(d)(e)$	x_1^5
π_2	72^0 Rotation	$(abcde)$	x_5
π_3	144^0 Rotation	$(acebd)$	x_5
π_4	216^0 Rotation	$(adbec)$	x_5
π_5	288^0 Rotation	$(aedcb)$	x_5
π_6	Flip around a	$(a)(be)(cd)$	$x_1x_2^2$
π_7	Flip around b	$(ac)(b)(de)$	$x_1x_2^2$
π_8	Flip around c	$(ae)(bd)(c)$	$x_1x_2^2$
π_9	Flip around d	$(ab)(ce)(d)$	$x_1x_2^2$
π_{10}	Flip around e	$(ad)(bc)(e)$	$x_1x_2^2$

So, using Theorem 1.4, we get the number of 2-colorings of a floating pentagon to be $\frac{1}{|G|}\sum_{\pi\epsilon G}\psi(\pi) = \frac{1}{10}(2^5 + 4(2) + 5(2^3)) = \frac{1}{10}(80) = 8$.

Table 2.2: Pattern Inventory of a Pentagon

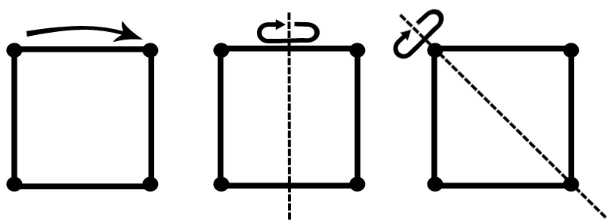
π_is	Polynomial	Expanded Polynomial
π_1	$(b+w)^5$	$b^5 + 5b^4w + 10b^3w^2 + 10b^2w^3 + 5bw^4 + w^5$
$\pi_2, \pi_3, \pi_4, \pi_5$	$(b^5 + w^5)$	$b^5 + w^5$
$\pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}$	$(b+w)(b^2+w^2)^2$	$b^5 + b^4w + 2b^3w^2 + 2b^2w^3 + bw^4 + w^4$

To find the pattern inventory of the black and white colorings of a floating pentagon, we find $P_G((b+w),(b^2+w^2),(b^3+w^3),(b^4+w^4),(b^5+w^5))$ by summing the third column in Table 2.2. From this we get $\frac{1}{10}(10b^5 + 10b^4w + 20b^3w^2 + 20b^2w^3 + 10bw^4 + 10w^5) = b^5 + b^4w + 2b^3w^2 + 2b^2w^3 + bw^4 + w^5$.

Application 2.2: Vertex and Edge cycles of a floating square

Consider a square, are the number of 2-colorings of the vertices the same as the number of 2-colorings of the edges? Let us begin with the vertices. There are three types of movements associated with the floating square. The first being the rotations, of which the floating square has four: $0^0, 90^0, 180^0$, and 270^0. The second type of rotation associated with the square is a "flip" around a line created when we join to opposing endpoints, there are two of these. Also, there are two of the third type, a "flip" around a line created by joining the midpoints of opposite edges. Figure 2.2 shows how the symmetries of the square work.

Figure 2.2: Symmetries of the Square



Let a, b, c, and d be the vertices. Then we get the vertex cycles of the Square in Table 2.3.

Table 2.3: Vertex Cycles of a Floating Square

Symmetry	Description	Vertex Cycles	Cycle Representation
π_1	Identity	$(a)(b)(c)(d)$	x_1^4
π_2	90^0 Rotation	$(abcd)$	x_4
π_3	180^0 Rotation	$(ac)(bd)$	x_2^2
π_4	270^0 Rotation	$(adcb)$	x_4
π_5	Flip Between a and c	$(a)(bd)(c)$	$x_1^2x_2$
π_6	Flip Between b and d	$(ac)(b)(d)$	$x_1^2x_2$
π_7	Flip Midpoints of ab and cd	$(ab)(cd)$	x_2^2
π_8	Flip Midpoints of ad and bc	$(ad)(bc)$	x_2^2

Before we move on to finding P_G for the 2-colorings of the vertices, let us take a look at the cycle representation of the edges. Label the edges e, f, g, and h. Then we have the following table.

Table 2.4: Edge Cycles of a Floating Square

Symmetry	Description	Edge Cycles	Cycle Representation
π_1	Identity	$(e)(f)(g)(h)$	x_1^4
π_2	90^0 Rotation	$(efgh)$	x_4
π_3	180^0 Rotation	$(eg)(fh)$	x_2^2
π_4	270^0 Rotation	$(ehgf)$	x_4
π_5	Flip Between a and c	$(eh)(fg)$	x_2^2
π_6	Flip Between b and d	$(ef)(gh)$	x_2^2
π_7	Flip Midpoints of e and g	$(e)(fh)(g)$	$x_1^2x_2$
π_8	Flip Midpoints of f and h	$(eg)(f)(h)$	$x_1^2x_2$

We first notice that the cycle representation for the rotations is the same for both the vertex and edge cycles. While the flips are different, we notice that if we replace the vertex flips with the edge flips in the edge cycle table, we have the same cycle representations. Thus, even before we calculate P_G, we know that we will have the same number of colorings. So, in both cases, $P_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4)$, and furthermore, $P_G(2, 2, 2, 2) = \frac{1}{8}(2^4 + 2(2^3) + 3(2^2) + 2(2)) = \frac{1}{8}(48) = 6$. We find that the pattern inventory will also be the same, and we work that out to be $b^4 + b^3w + 2b^2w^2 + bw^3 + b^4$.

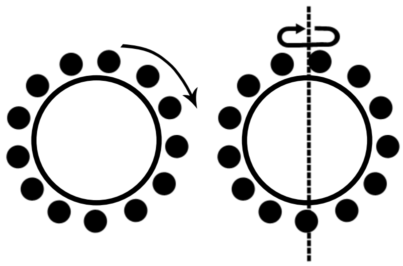
Application 2.3: Knights of the Round Table

Let us say that Arthur had thirteen knights including himself. Consider the number of ways he could paint the chairs around the round table if he only had access to black and white paint. We consider each of the chairs to be a vertex on a 2-dimensional object with thirteen vertices. As thirteen is prime, having no divisors, this shape will have 12 non-identity rotations, each of which has a cycle structure representation of x_{13}. It also will have 13 "flips" around a line from a vertex to the midpoint of it's opposing edge. Each of these "flips" has the cycle structure representation $x_1x_2^6$. Let us consider the identity rotation π_1 to be the 0^0 rotation of the object which has the cycle structure representation x_1^{13}. Furthermore, let $\pi_2, \pi_3, ..., \pi_{13}$ be the other rotations of the object(the $(\frac{360}{13})^0, (\frac{2(360)}{13})^0, ..., (\frac{12(360)}{13})^0$ rotations), and let $\pi_{14}, \pi_{15}, ..., \pi_{26}$ be the "flips" of the object. To find $P_G(x_1, x_2, ..., x_{13})$, we look at Table 2.5. We can see the rotations in Figure 2.3.

Table 2.5: Cycle Structure Representation of an Object with Thirteen Vertices

Symmetry	**Cycle Structure Representation**	no. of Axes
Identity	x_1^{13}	1
Rotations	x_{13}	12
Flips	$x_1x_2^6$	13

Figure 2.3: Symmetries of the 13-gon



So, $P_G(x_1, x_2, ...x_{13}) = \frac{1}{26}(x_1^{13} + 13x_1x_2^6 + 12x_{13})$. We find the number of 2-colorings by substituting 2 in for each x_i, so the number of colorings is equal to $P_G(2, 2, ..., 2) = \frac{1}{26}(2^{13} + 13(2)(2^6) + 12(2)) = \frac{1}{26}(9880) = 380$. Similarly we find the pattern inventory to be $b^{13} + b^{12}w + 6b^{11}w^2 + 14b^{10}w^3 + 35b^9w^4 + 57b^8w^5 + 76b^7w^6 + 76b^6w^7 + 57b^5w^8 + 35b^4w^9 + 14b^3w^{10} + 6b^2w^{11} + bw^{12} + w^{13}$.

Application 2.4: C_n A Necklace of length n

A necklace is a number of objects strung together by a bound chain, or a chain whose beginning is tied to its end. We wish to investigate the number of different ways to color such necklaces. We can note that if we arrange these objects uniformly along the chain, and connect each object to the next object with a single edge, we have a regular n-gon. We keep this fact in mind as we find the symmetries of the necklace. The Identity rotation would leave all of the objects of the necklace fixed. We would have $n - 1$ rotations which would move all of the objects in $\frac{n}{gcd(i,n)}$-cycles, where i is the number of places each object moves over.

There are also flips of the necklace, which depend on whether n is odd or even. If n is odd, then there are n flips which occur around axes created from each vertex to the midpoint of the opposite edge. Each of these flips keeps one vertex fixed, while moving the rest in 2-cycles. If n is even, then there are two types of flips. There are $\frac{n}{2}$ flips of the first type, which are around an axis created by joining opposite vertices. Each of these flips keeps two vertices fixed, while moving the rest in 2-cycles. There are also $\frac{n}{2}$ flips of the second type, which are flips around an axis created by joining the midpoints of opposite edges. These flips leave no vertex fixed, and rotate all of them in 2-cycles. Table 2.6 gathers this information in a more clear way.

Table 2.6: Cycle Structure Notation of C_n

Symmetry	n odd	n even	No. of Axes
Identity	x_1^n	x_1^n	1
Rotation i	$x_{\frac{n}{gcd(i,n)}}^{gcd(i,n)}$	$x_{\frac{n}{gcd(i,n)}}^{gcd(i,n)}$	$n-1$
Vertex to Midpoint flip	$x_1x_2^{\frac{n-1}{2}}$	None	n
Vertex to Vertex flip	None	$x_1^2x_2^{\frac{n-2}{2}}$	$\frac{n}{2}$
Midpoint to Midpoint flip	None	$x_2^{\frac{n}{2}}$	$\frac{n}{2}$

Since we seek the number of colorings of each of these types of necklaces, we can find such by summing the appropriate columns of Table 2.6. If we sum column 2, we find $P_V(x_1, x_2, ..., x_n) = \frac{1}{2n}(x_1^n + nx_1x_2^{\frac{n-1}{2}} + \sum_{i=1}^{n-1} x_{\frac{n}{gcd(i,n)}}^{gcd(i,n)})$ for n odd. We can find the pattern inventory of the 2-coloring of such necklaces by finding:

$$P_V(b+w, b^2+w^2, ..., b^n+w^n) = \frac{1}{2n}((b+w)^n + n(b+w)(b^2+w^2)^{\frac{n-1}{2}} + \sum_{i=1}^{n-1}(b^{\frac{n}{gcd(i,n)}} + w^{\frac{n}{gcd(i,n)}})^{gcd(i,n)}).$$

Conversely, if we sum column 3 of Table 2.15, we get

$$P_V(x_1, x_2, ..., x_n) = \frac{1}{2n}(x_1^n + \frac{n}{2}x_1^2x_2^{\frac{n-2}{2}} + \frac{n}{2}x_2^{\frac{n}{2}} + \sum_{i=1}^{n-1} x_{\frac{n}{gcd(i,n)}}^{gcd(i,n)})$$

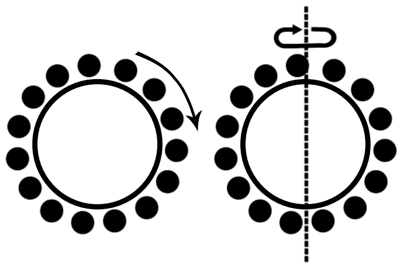
for n even. We can also find the pattern inventory of the 2-coloring of such necklaces by finding $P_V(b+w, b^2+w^2, ..., b^n+w^n)$ which is,

$$\frac{1}{2n}((b+w)^n + \frac{n}{2}(b+w)^2(b^2+w^2)^{\frac{n-2}{2}} + \frac{n}{2}(b^2+w^2)^{\frac{n}{2}} + \sum_{i=1}^{n-1}(b^{\frac{n}{gcd(i,n)}} + w^{\frac{n}{gcd(i,n)}})^{gcd(i,n)})$$

.

Now, suppose The UN Security council is attempting to make up it's membership, and that there are fifteen nations whom sit on this council. Each nation selected get to choose whether to drink water or soda at any particular meeting. How many different ways are there to arrange the drinks at a round table. This problem can be broken up into a necklace of length fifteen. Figure 2.4 illustrates this.

Figure 2.4: Symmetries of a 15-gon



Since our $n = 15$ we can use our first formula. Notice that for i = 3, 6, 9, and 12 the cycle structure notation for the rotations will be x_5^3, and that for i = 5 and 10, the cycle structure notation for the rotations will be x_3^5. The remaining eight rotations will have the cycle structure notation x_{15}. So, we find $P_V(x_1, x_2, ..., x_{15}) = \frac{1}{30}(x_1^{15} + 15x_1x_2^7 + 2x_3^5 + 4x_5^3 + 8x_{15})$. We find the number of 2-colorings of this necklace to be $P_V(2, 2, ..., 2) = \frac{1}{30}(2^{15} + 15(2^8) + 2(2^5) + 4(2^3) + 8(2)) = \frac{1}{30}(36720) = 1224$. So, there are 1224 ways to arrange the table with drinks.

If we wished to find the pattern inventory of these 1224 colorings, we would find $P_V(b + w, b^2 + w^2, ..., b^{15} + w^{15}) = \frac{1}{30}((b + w)^{15} + 15(b + w)(b^2 + w^2)^7 + 2(b^3 + w^3)^5 + 4(b^5 + w^5)^3 + 8(b^{15} + w^{15})) = b^{15} + b^{14}w + 7b^{13}w^2 + 19b^{12}w^3 + 56b^{11}w^4 + 111b^{10}w^5 + 185b^9w^6 + 232b^8w^7 + 232b^7w^8 + 185b^6w^9 + 111b^5w^{10} + 56b^4w^{11} + 19b^3w^{12} + 7b^2w^{13} + bw^{14} + w^{15}$.

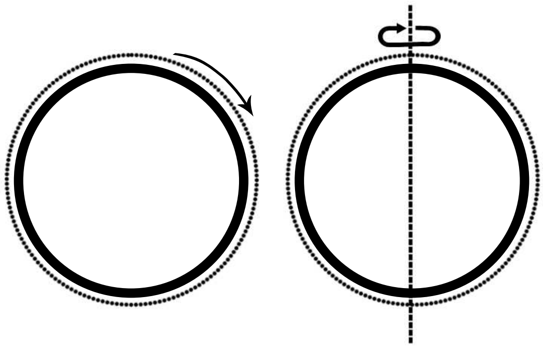
Now, suppose the UN Security Council had only fourteen members. Then, the necklace would be of length 14, so we would use the second formula. For i = 2,4,6,8,10, or 12 the cycle structure notation would be x_7^2 and for i = 7, the cycle structure notation would be x_2^7. The remaining six rotations would have cycle structure notation x_{14}. Thus, we find our $P_V(x_1, x_2, ..., x_3) = \frac{1}{28}(x_1^{14} + 7x_1^2x_2^6 + 7x_2^7 + x_2^7 + 6x_7^2 + 6x_{14})$ and $P_V(2, 2, ..., 2) = \frac{1}{28}(2^{14} + 7(2^8) + 8(2^7) + 6(2^2) + 6(2)) = \frac{1}{28}(19236) = 687$. So there are 687 such arrangements of drinks.

Again, we wish to find the pattern inventory of these arrangements, so we find $P_V(b + w, b^2 + w^2, ..., b^{14} + w^{14}) = \frac{1}{28}((b + w)^{14} + 7(b + w)^2(b^2 + w^2)^6 + 8(b^2 + w^2)^7 + 6(b^7 + w^7)^2 + 6(b^{14} + w^{14})) = b^{14} + b^{13}w + 7b^{12}w^2 + 16b^{11}w^3 + 47b^{10}w^4 + 79b^9w^5 + 126b^8w^6 + 133b^7w^7 +$

$126b^6w^8 + 79b^5w^9 + 47b^4w^{10} + 16b^3w^{11} + 7b^2w^{12} + bw^{13} + w^{14}$.

Now suppose the olympics has 163 nations participating this year and the olympic committee is developing an opening ceremony. They want all 163 nations to participate and will give each nation the option of bringing a banner or a flag representing their home nation to the ceremony. If these banners and flags are arranged around the olympic torch, we seek to find the number of arrangements of banners and flags that is possible. Notice that this can be broken down into a necklace of length 163. Also, notice 163 is prime, so all 162 rotations have the cycle structure notation x_{163}. Then, from our first formula we get $P_V(x_1, x_2, ..., x_{163}) = \frac{1}{326}(x_1^{163} + 163x_1x_2^{81} + 162x_{163})$. To find the number of arrangement, we find the number of colorings of this necklace. We get $P_V(2, 2, ..., 2) = \frac{1}{326}(2^{163} + 163(2^{82}) + 162(2))$. Notice that this will turn out to be a very large number, as 2^{163} is a fifty digit number itself. Figure 2.5 shows the symmetries of this polygon.

Figure 2.5: Symmetries of a 163-gon



Application 2.5: S_n, the Star Graph with $n + 1$ vertices

Let S_n be the star graph with $n + 1$ vertices. We seek a formula to find the number of 2-colorings of the vertices of the star graph. If we evenly space the outer vertices on the star graph, we can consider the number of 2-colorings of the vertices of a star graph to be $2P_V(2, 2, ..., 2)$ where P_V is the number of two colorings of a regular n-gon. For any regular n-gon, there will be n rotations of the figure, the first of which being the identity, and the subsequent ones being a rotation of the figure by $(\frac{360i}{n})^0)$ for $i = 1, 2, ..., n-1$. Each of these rotations will cycle structure notation that depends on the largest common divisor of n and i, we find that cycle structure notation to be $x_{\frac{n}{lcd(i,n)}}^{lcd(i,n)}$. We find the cycle structure notation of the identity rotation to be x_1^n.

If this n-gon is free to float in space, we find that the type of flips depends on whether n is even or odd. If n is odd, then there are n flips which flip around an axis created by connecting a vertex with the midpoint of the opposite edge. These flips leave only vertex fixed and have cycle structure notation $x_1x_2^{\frac{n-1}{2}}$. If n is even, then there are two types of flips. The first type of flip is around an axis created by connecting two opposite endpoints. There are $\dfrac{n}{2}$ of these flips, and as they leave two vertices fixed we find their cycle structure notation to be $x_1^2x_2^{\frac{n-2}{2}}$. The other type of flip is one around an axis created by joining the midpoints of opposite edges. This type of flip leaves no vertices fixed, so we find the cycle structure notation to be $x_2^{\frac{n}{2}}$. There are $\dfrac{n}{2}$ of these rotations.

There are n rotations and n flips, so $|G| = 2n$. Thus, using Theorem 1.4, we find that $P_V(x_1, x_2, ..., x_{2n}) = \frac{1}{2n}(x_1^n + nx_1x_2^{\frac{n-1}{2}} + \sum_{i=1}^{n-1} x_{\frac{n}{lcd(i,n)}}^{lcd(i,n)})$ for n odd. Thus, for n odd, $2P_V(x_1, x_2, ..., x_{2n}) = \frac{1}{n}(nx_1x_2^{\frac{n-1}{2}} + \sum_{i=1}^{n} x_{\frac{n}{lcd(i,n)}}^{lcd(i,n)})$. So, the number of 2-colorings of the Star graph with $n + 1$ vertices is $2P_V(2, 2, ..., 2) = \frac{1}{n}(n(2^{\frac{n+1}{2}}) + \sum_{i=1}^{n} 2^{lcd(i,n)})$ when n is odd.

If n is even, then $2P_V(x_1, x_2, ..., x_{2n}) = \frac{1}{n}(\frac{n}{2}(x_1^2x_2^{\frac{n-1}{2}}) + \frac{n}{2}(x_2^{\frac{n}{2}}) + \sum_{i=1}^{n} x_{\frac{n}{lcd(i,n)}}^{lcd(i,n)})$. So, if n is even, the number of 2-colorings of S_n is $2P_V(2, 2, ..., 2) = \frac{1}{n}(\frac{n}{2}(2^{\frac{n+1}{2}}) + \frac{n}{2}(2^{\frac{n}{2}}) + \sum_{i=1}^{n} 2^{lcd(i,n)})$. Thus we have the formulas we sought. The pattern inventory of such a graph would be obtained by finding the $2n$th degree polynomial $P_V(b + w, b^2 + w^2, ..., b^{2n} + w^{2n})$.

At this point, we could easily apply a similar formula to K_n, the complete graph with n vertices. To find the number of 2-colorings of the vertices of K_n we would simply find $P_V(2, 2, ..., 2)$ using the formula we derived above.

Application 2.6: $K_{m,n}$, the Complete Bipartite Graph with $m + n$ vertices

To find the general case, we know that there will be a number of cycles equal to $n! \times m!$, so we will concentrate on a small example, so that the computations are not overly difficult. Let $m = 2$ and $n = 3$, then we have the graph in Figure 2.6.

We can switch the vertices on the left, or any two of the vertices on the right. This gives us $2! \times 3! = 2 \times 6 = 12$ cycles. The identity is where we do not move any of the vertices. Let us label the vertices on the right a and b, and the vertices on the left d, e, and f. Then we have the following chart of the cycle structures.

Now, imagine that this graph represents two transmitting nodes and three receiving nodes. At any time, any of these nodes can either be on or off. We seek to find the number of combinations of on and off patterns that is possible. To this end, we first find

$P_V(x_1, x_2, ..., x_5)$. We find this by summing columns 3 and 6 from Table 2.7. From this, we get $P_V(x_1, x_2, ..., x_5) = \frac{1}{12}(x_1^5 + 4x_1^3x_2 + 2x_1^2x_3 + 3x_1x_2^2 + 2x_2x_3)$. Now, to find the 2-colorings, we find $P_V(2, 2, ..., 2) = \frac{1}{12}(2^5 + 4(2^4) + 2(2^3) + 3(2^3) + 2(2^2)) = \frac{1}{12}(144) = 12$. We can find the pattern inventory of the 12 colorings we found by finding $P_V(b+w, b^2+w^2, ..., b^5+w^5) = \frac{1}{12}((b+w)^5+4(b+w)^3(b^2+w^2)+2(b+w)^2(b^3+w^3)+3(b+w)(b^2+w^2)^2+2(b^2+w^2)(b^3+w^3)) = \frac{1}{12}(12b^5+24b^4w+36b^3w^2+36b^2w^3+24bw^4+12w^5) = b^5+2b^4w+3b^3w^2+3b^2w^3+2bw^4+w^5$.

Figure 2.6: Complete Bipartite Graph

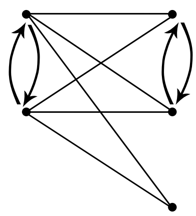


Table 2.7: Cycles and their Cycle Structure Notation of the Complete Bipartite Graph

Symmetry	Cycles	Cycle Structure	Symmetry	Cycles	Cycle Structure
π_1	$(a)(b)(d)(e)(f)$	x_1^5	π_7	$(ab)(d)(e)(f)$	$x_1^3x_2$
π_2	$(a)(b)(de)(f)$	$x_1^3x_2$	π_8	$(ab)(de)(f)$	$x_1x_2^2$
π_3	$(a)(b)(df)(e)$	$x_1^3x_2$	π_9	$(ab)(df)(e)$	$x_1x_2^2$
π_4	$(a)(b)(d)(ef)$	$x_1^3x_2$	π_{10}	$(ab)(d)(ef)$	$x_1x_2^2$
π_5	$(a)(b)(def)$	$x_1^2x_3$	π_{11}	$(ab)(def)$	x_2x_3
π_6	$(a)(b)(dfe)$	$x_1^2x_3$	π_{12}	$(ab)(dfe)$	x_2x_3

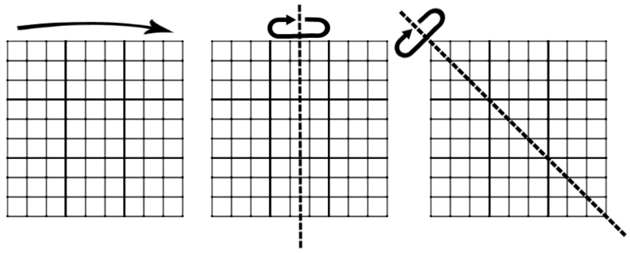
Application 2.7: Cycles of a 9×9 Grid

Consider a 9×9 grid, often used for number games like Sodoku. We consider a vertex to be the point at which two lines on the graph intersect, edges to be the pieces of lines between vertices, and faces to be the open areas enclosed by edges. Using this method, the grid has 100 vertices, 180 edges, and 81 faces. Like the square, the grid has three types of symmetries. The four rotations (including the identity), the two "flips" around opposite outer endpoints, and the two "flips" around opposite midpoints of the outer square. Let us look at the number of 2-colorings of the vertices, edges, and faces of the grid. Figure 2.7 shows how these symmetries work on the 9×9 grid.

First consider the set of vertices of the 9×9 grid. The identity rotation leaves all vertices fixed, thus has a cycle structure representation of x_1^{100}. The 180^0 rotation leaves none of the vertices fixed, and rotates them in such a way so that all the vertices move in 2-cycles.

Thus, the 180^0 rotation has a cycle structure notation of x_2^{50}. The remaining rotations leave no vertices fixed, and move the vertices in 4-cycles. So the 90^0 and 270^0 rotations have cycle structure representations of x_4^{25}. A flip around the opposite, outer vertices leaves all ten vertices lying along the axis created, while moving the remaining vertices in a 2-cycle, so both vertex flips have the cycle structure representation of $x_1^{10}x_2^{45}$. As the axes created by the outer midpoint flip does not pass through any vertices, no vertices stay fixed during those motions. So, all the vertices move in 2-cycles, leaving the cycle structure representation x_2^{50} for each. Figure 2.7 gives us an idea of what these symmetries might do to a 9×9 grid.

Figure 2.7: Symmetries of a 9×9 grid



Now consider the edges. As with the vertices, all of the edges remain fixed for the identity, the 180^0 rotation moves all the edges in 2-cycles, and the other rotations move the edges in 4-cycles. This gives these rotations cycle structure notations of x_1^{180}, x_2^{90}, and x_4^{45} respectively. As for the midpoint, and endpoint flips, how they affect the edges is opposite of how they affected the vertices. The endpoint flip does not pass through any edges, so it moves all of them in 2-cycles, having a cycle structure representation of x_2^{90}. The midpoint flip, on the other hand, passes through ten edges, leaving them fixed, while moving the rest of the edges in 2-cycles. Thus, the midpoint flip has cycle structure representation $x_1^{10}x_2^{85}$.

Finally, consider the faces of the grid. The identity rotation still leaves all of the faces fixed, and has a cycle structure representation of x_1^{81}. Since this is an odd grid, there is one square in the middle that will not rotate with the rest when the grid is rotated and thus remains fixed. For the remaining faces, the rotations move them as they did the edges and vertices. So the 180^0 rotation has the cycle structure representation $x_1x_2^{40}$ and the other rotations have the cycle structure representation $x_1x_4^{20}$. All of the flips have axes that pass through nine faces, leaving them fixed, and moving the rest in 2-cycles. So all of them have the cycle structure representation $x_1^9x_2^{36}$. Since this is a lot of information to process, Table 2.8 will keep things in order for us.

So, if we let V be the set of vertices, E be the set of edges, and F be the set of faces, we want to find P_V, P_E, and P_F. If we sum Column 2 in Table 2.8, we find $P_F(x_1, x_2, ..., x_{81}) = \frac{1}{8}(x_1^{81} + 4x_1^9x_2^{36} + x_1x_2^{40} + 2x_1x_4^{20})$, which means that $P_F(2, 2, ..., 2) = \frac{1}{8}(2^{81} + 4(2^{45}) + 2^{41} + 2(2^{21})) = \frac{1}{8}(2417851639372194865217536) = 302231454921524358152192$. It's clear that

P_V and P_E will be much larger than this, so we will write out $P_V(x_1, x_2, ..., x_{100})$ and $P_E(x_1, x_2, ..., x_{180})$, leaving the calculations out. By summing Column 4 of Table 2.8, we find $P_V(x_1, x_2, ..., X_{100}) = \frac{1}{8}(x_1^{100} + 2x_1^{10}x_2^{45} + 3x_2^{50} + 2x_4^{25})$, and by summing Column 3, we find $P_E(x_1, x_2, ..., x_{180}) = \frac{1}{8}(x_1^{180} + 2x_1^{10}x_2^{85} + 3x_2^{90} + 2x_4^{45})$.

Table 2.8: Cycle Notation of a 9×9 Grid
v = 100, e = 180, f = 81

Rotation	Face Cycles	Edge Cycles	Vertex Cycles	no. of Axes
Id	x_1^{81}	x_1^{180}	x_1^{100}	1
$90^0 Rot$	$x_4^{20}\ x_1^1$	x_4^{45}	x_4^{25}	1
$180^0 Rot$	$x_2^{40}\ x_1^1$	x_2^{90}	x_2^{50}	1
$270^0 Rot$	$x_4^{20}\ x_1^1$	x_4^{45}	x_4^{25}	1
Vertical Flip	$x_2^{36}\ x_1^9$	$x_2^{85}\ x_1^{10}$	x_2^{50}	1
Horizontal Flip	$x_2^{36}\ x_1^9$	$x_2^{85}\ x_1^{10}$	x_2^{50}	1
Endpoint Flip	$x_2^{36}\ x_1^9$	x_2^{90}	$x_2^{45}\ x_1^{10}$	2

If we wished to find the pattern inventory of the 2-colorings of the faces of the 9×9 grid, we would find $P_F(b + w, b^2 + w^2, ..., b^{81} + w^{81}) = \frac{1}{8}((b + w)^{81} + 2(b + w)(b^4 + w^4)^{20} + (b + w)(b^2 + w^2)^{40} + 4(b + w)^9(b^2 + w^2)^{36})$. This is clearly a very long polynomial, the pattern inventory of the vertices and edges is even larger, having terms to the 100th and 180th power respectively.

Application 2.8: Colorings of a Flattened Octahedron

Consider an octahedron, which is a 3-dimensional object that glues together eight triangles. If we flatten this object, we now have the 2-dimensional object shown below. We can use a similar method to find the number of colorings of the vertices of this object.

We can treat this 2-dimensional object like a triangle, with some extra vertices used, and because of this we have two types of symmetries that can act on the object. The first type are the rotations. We can rotate this object 120^0 and 240^0, each of which moves all six vertices in 3-cycles. The other symmetry that can act on the object is flips around an axis. these axes are created from each of the outer vertices, and pass through the opposing inner vertex on the way to the midpoint of the opposite edge. Each of these three axes leaves two vertices fixed (the two it passes through), and rotates the remaining vertices in 2-cycles. There is also the identity symmetry which does not move any of the vertices. Table 2.9 organizes these symmetries into a more easily readable form.

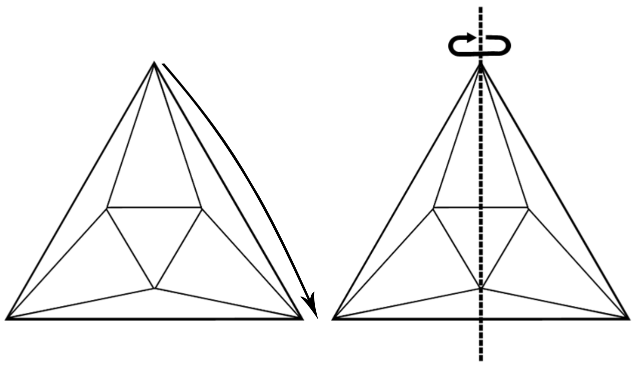
Now, if we wish to find the number of 2-colorings of the vertices if this object is free to float in space, we could sum Column 2 of Table 2.9 to find $P_V(x_1, x_2, ..., x_6) = \frac{1}{6}(x_1^6 + 3x_1^2x_2^2 + 2x_3^2)$. Then the number of 2-colorings would be equal to $P_V(2, 2, ..., 2) = \frac{1}{6}(2^6 + 3(2^4) + 2(2^2)) =$

$\frac{1}{6}(120) = 20$. We could additionally find the pattern inventory by finding $P_V(b+w, b^2+w^2, ..., b^6+w^6) = \frac{1}{6}((b+w)^6 + 3(b+w)^2(b^2+w^2)^2 + 2(b^3+w^3)^2) = \frac{1}{6}(6b^6 + 12b^5w + 24b^4w^2 + 36b^3w^3 + 24b^2w^4 + 12bw^5 + 6w^6) = b^6 + 2b^5w + 4b^4w^2 + 6b^3w^3 + 4b^2w^4 + 2bw^5 + w^6$.

Table 2.9: Cycle Structure Notation of a Flattened Octahedron

Symmetry	Cycle Structure Notation	No. of Axes
Identity	x_1^6	1
120^0 Rotation	x_3^2	1
240^0 Rotation	x_3^2	1
Flip	$x_1^2x_2^2$	3

Figure 2.8: Symmetries of a Flattened Octahedron



Now, suppose this object was the layout of an airport, where each vertex is a light to signal the airplanes. Now, say each light has a possible of three colors (red, yellow, and green) to tell planes where they can land. We seek to find the number of ways this airport could be lit up. Notice, since we cannot flip the airport, we will only be counting the rotations. To find $P_V(x_1, x_2, ..., x_6)$ we sum the first three rows of Column 2 in Table 2.9. This gives us $P_V(x_1, x_2, ..., x_6) = \frac{1}{3}(x_1^6 + 2x_3^2)$, and the number of 3-colorings of the object can be found by $P_V(3, 3, ..., 3) = \frac{1}{3}(3^6 + 2(3^2)) = \frac{1}{3}(747) = 249$. So, there are 249 ways the airport could be lit up. The pattern inventory would be found by finding $P_V(r+y+g, r^2+y^2+g^2, ..., r^6+y^6+g^6) = \frac{1}{3}((r+y+g)^6 + 2(r^3+y^3+g^3)^2)$.

3 3-Dimensional Applications

Application 3.1: Colorings of a Tetrahedron

Returning to three dimensions, let us examine how this theorem would apply to the 2-colorings of the edges, faces, and vertices of a regular tetrahedron, which is four equilateral triangles arranged in 3-dimensions so that each triangle is connected to all others by one edge. So, we have four faces, four vertices, and six edges. The tetrahedron has two types of rotations, the first one, which we will call a $T1$ rotation, rotates around an axis that spans from a vertex to the exact center of the opposite face. A $T1$ rotation can rotate the tetrahedron 120^0 or 240^0. The second type of rotation, which we will call a $T2$ rotation, rotates the tetrahedron around an axis which spans the midpoints of opposite edges. The $T2$ rotation rotates the tetrahedron 180^0. Furthermore, there is an identity rotation, which leaves all faces, vertices, and edges fixed. We know that the cycle structure representation for the faces, vertices, and edges of the identity will be x_1^4, x_1^4, and x_1^6 respectively.

To find the cycle structure representations, let us label the faces a, b, c, and d, the vertices u, v, w, and y, and the edges o, p, q, r, s, and t. Now, we examine a $120^0 T1$ rotation, which rotates the tetrahedron around a line from u to the center of d. We find that the face cycles created by this rotation are $(abc)(d)$, the vertex cycles are $(u)(vwy)$, and the edge cycles are $(opq)(rst)$. So the face, vertex, and edge cycle structure representation will be x_1x_3, x_1x_3, and x_3^2 respectively. The other $T1$ rotations will follow a similar pattern, and thus have the same cycle structure representations. Let us now look at the $T2$ rotation which rotates the tetrahedron around the line created from the midpoint of q to the midpoint of r. The face cycles created by this rotation are $(ad)(bc)$, the vertex cycles created by this rotation are $(uy)(vw)$, and the edge cycles created by this rotation are $(os)(pt)(q)(r)$. Thus, the face, vertex, and edge cycle structure representation will be x_2^2, x_2^2, and $x_1^2x_2^2$ respectively. The other $T2$ rotations will follow a similar pattern, and have the same cycle structure representation. Table 3.1 shows the cycle structure representations for the faces, edges, and vertices, while Figure 3.1 shows us how the symmetries work on the tetrahedron.

Figure 3.1: Symmetries of a Tetrahedron

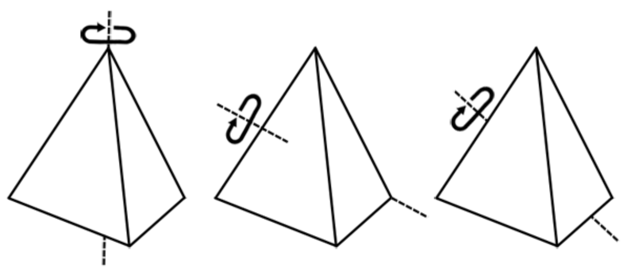


Table 3.1: Cycle Notation of a Tetrahedron
v = 4, e = 6, f = 4

Rotation	Face Cycles	Vertex Cycles	Edge Cycles	no. of Axes
Id	x_1^4	x_1^4	x_1^6	1
$120^0 T1$	$x_3\ x_1$	$x_3\ x_1$	x_3^2	4
$240^0 T1$	$x_3\ x_1$	$x_3\ x_1$	x_3^2	4
$180^0 T2$	x_2^2	x_2^2	$x_2^2\ x_1^2$	3

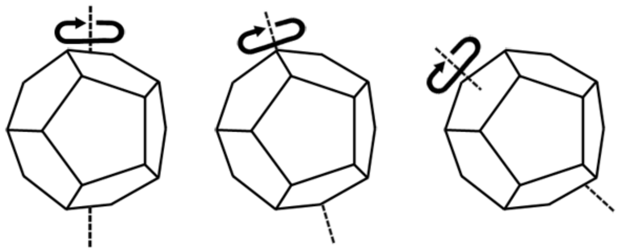
By comparison, we can see that $P_F = P_V$, so we can sum either Column 2, or Column 3 of Table 3.1 to find the face and vertex colorings. Doing this, we find $P_F(x_1, x_2, x_3, x_4) = P_V(x_1, x_2, x_3, x_4) = \frac{1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2)$, so this means the number of face or vertex colorings is $P_F(2,2,2,2) = P_V(2,2,2,2) = \frac{1}{12}(2^4 + 8(2^2) + 3(2^2) = \frac{1}{12}(60) = 5$. Similarly we can find the number of edge colorings by summing Column 4 in Table 3.1 to get $P_E(x_1, x_2, ..., x_6) = \frac{1}{12}(x_1^6 + 8x_3^2 + 3x_1^2x_2^2)$. So the number of edge colorings of the tetrahedron is $P_E(2,2,2,2,2,2) = \frac{1}{12}(2^6 + 8(2^2) + 3(2^4)) = \frac{1}{12}(144) = 12$.

To find the pattern inventory of the 2-colorings of the faces or vertices of the tetrahedron, we would find $P_F(b+w, b^2+w^2, b^3+w^3, b^4+w^4) = \frac{1}{12}((b+w)^4 + 8(b+w)(b^3+w^3) + 3(b^2+w^2)^2) = \frac{1}{12}(12b^4 + 12b^3w + 12b^2w^2 + 12bw^3 + 12w^4) = b^4 + b^3w + b^2w^2 + bw^3 + w^4$. We could also find the pattern inventory of the 2-colorings of the edges of the tetrahedron by finding $P_E(b+w, b^2+w^2, ..., b^6+w^6) = \frac{1}{12}((b+w)^6 + 8(b^3+w^3)^2 + 3(b+w)^2(b^2+w^2)^2) = \frac{1}{12}(12b^6 + 12b^5w + 24b^4w^2 + 48b^3w^3 + 24b^2w^4 + 12bw^5 + 12w^6) = b^6 + b^5w + 2b^4w^2 + 4b^3w^3 + 2b^2w^4 + bw^5 + w^6$.

Application 3.2 Colorings of a Dodecahedron

Consider a regular dodecahedron, which is twelve copies of a pentagon formed into a three dimensional shape. Again, let E be the set of edges of the dodecahedron, F be the set of faces of the dodecahedron, and V be the set of vertices of the dodecahedron. We seek to find the number of ways to color the edges, vertices, or faces with two colors. Note that the dodecahedron has twelve faces, twenty vertices, and thirty edges. The dodecahedron has three types of rotations. The first type, which we will call $T1$, of rotations rotates the dodecahedron around an axis formed by joining the centers of opposite faces. $T1$ rotations can rotate the dodecahedron $72^0, 144^0, 216^0$, and 288^0. The second type of rotation, $T2$ rotation, rotates the dodecahedron around an axis created by connecting two opposite vertices. $T2$ rotations can rotate the dodecahedron 120^0 and 240^0. The last type of rotation, called $T3$ rotations, rotate the dodecahedron around an axis created by connecting midpoints of opposite edges. The $T3$ rotations rotate the dodecahedron 180^0. Again, we consider the identity rotation to be the one that leaves the faces, edges, and vertices fixed, and thus has cycle structure representations of x_1^{12} for the faces, x_1^{20} for the vertices, and x_1^{30} for the edges. Figure 3.2 shows how the symmetries work on the Dodecahedron.

Figure 3.2: Symmetries of a Dodecahedron



Consider the $T1$ rotations. When the $T1$ rotations rotate the dodecahedron, the two faces used to create that specific rotation remain fixed. All other faces, as well as all the vertices and edges, will move about the axis in 5-cycles. Thus, all $T1$ rotations have a face cycle structure representation of $x_1^2x_5^2$, edge cycle structure representations x_5^6, and vertex structure representations x_5^4. Now consider the $T2$ rotations. These rotations leave the two vertices used in the axis creation fixed, and rotate all other vertices, as well as edges and faces, in 3-cycles. Thus, all of the $T2$ rotations have face cycle structure representations x_3^4, edge cycle structure representations x_3^{10}, and vertex cycle structure representations $x_1^2x_3^6$. Finally, consider the $T3$ rotations, which leave the two edges used in their creation fixed, rotating the other edges, as well as the vertices and faces, in 2-cycles. So, all $T3$ rotations have face cycle structure representations x_2^6, edge cycle structure representations $x_1^2x_2^{14}$, and vertex cycle structure representations x_2^{10}. Table 3.2 organizes these cycle structure representations.

Table 3.2: Cycle Structure Representations of a Dodecahedron

v = 20, e = 30, f = 12

Rotation	**Face Cycles**	**Vertex Cycles**	**Edge Cycles**	**no. of Axes**
Id	x_1^{12}	x_1^{20}	x_1^{30}	1
72^0T1	$x_5^2\ x_1^2$	x_5^4	x_5^6	6
144^0T1	$x_5^2\ x_1^2$	x_5^4	x_5^6	6
216^0T1	$x_5^2\ x_1^2$	x_5^4	x_5^6	6
288^0T1	$x_5^2\ x_1^2$	x_5^4	x_5^6	6
120^0T2	x_3^4	$x_3^6\ x_1^2$	x_3^{10}	10
240^0T2	x_3^4	$x_3^6\ x_1^2$	x_3^{10}	10
180^0T3	x_2^6	x_2^{10}	$x_2^{14}\ x_1^2$	15

If we sum Column 2 of Table 3.2, we find that $P_F(x_1, x_2, ..., x_{12}) = \frac{1}{60}(x_1^{12} + 24x_1^2x_5^2 + 20x_3^4 + 15x_2^6)$. So the number of 2-colorings of the edges of a dodecahedron is $P_F(2, 2, ..., 2) = \frac{1}{60}(2^{12} + 44(2^4) + 15(2^6)) = \frac{1}{60}(5760) = 96$. Summing Column 3 of Table 3.2 yield that

$P_V(x_1, x_2, ..., x_{20}) = \frac{1}{60}(x_1^{20} + 24x_5^4 + 20x_1^2x_3^6 + 15x_2^{10})$. So the number of 2-colorings of the vertices of a dodecahedron is $P_V(2, 2, ..., 2) = \frac{1}{60}(2^{20} + 24(2^4) + 20(2^8) + 15(2^{10})) = \frac{1}{60}(1069440) = 17824$. Finally, if we sum Column 4 of Table 3.2, we get that $P_E(x_1, x_2, ..., x_{30}) = \frac{1}{60}(x_1^{30} + 24x_5^6 + 20x_3^{10} + 15x_1^2x_2^{14})$. So, the number of 2-colorings of the edges of a dodecahedron is $P_E(2, 2, ..., 2) = \frac{1}{60}(2^{30} + 24(2^6) + 20(2^{10}) + 15(2^{16})) = \frac{1}{60}(1074746880) = 17912448$.

If we wished to find the pattern inventory of the 2-colorings of the faces of the dodecahedron, we would find $P_F(b + w, b^2 + w^2, ..., b^{12} + w^{12}) = \frac{1}{60}((b + w)^{12} + 24(b + w)^2(b^5 + w^5)^2 + 20(b^3 + w^3)^4 + 15(b^2 + w^2)^6) = b^{12} + b^{11}w + 3b^{10}w^2 + 5b^9w^3 + 12b^8w^4 + 14b^7w^5 + 24b^6w^6 + 14b^5w^7 + 12b^4w^8 + 5b^3w^9 + 3b^2w^{10} + bw^{11} + w^{12}$. Note that the pattern inventory of the vertices and edges would be much larger of a polynomial than this.

Application 3.3: Colorings of a Icosahedron

Suppose we have a icosahedron, a 3-dimensional object created by joining together twenty triangles, that's free to float in space. We seek a formula to find the number of 3-colorings of the faces, vertices and edges of such a figure. Note that there are twenty faces, twelve vertices, and thirty edges. We begin by examining the symmetries of the icosahedron. As usual, the identity rotation is one where none of faces, edges or vertices are moved from their original orientation. The first type of rotation (called a $T1$ rotation) occurs around an axis which connects the centers of two opposite faces. This type of rotation rotates the icosahedron 120^0, and 240^0. This rotation leaves two faces fixed, while rotating the remaining eighteen faces in 3-cycles. It leaves no vertices or edges fixed, and moves them all in 3-cycles. There are ten axes of the first type that the icosahedron can rotate around.

The second type of rotation (called a $T2$ rotation) is one around an axis which is created by joining opposite vertices. This type of rotation rotates the icosahedron $72^0, 144^0, 216^0$, and 288^0. This rotation leaves two vertices fixed, rotating the remaining ten in 5-cycles, likewise it rotates all faces and edges in 5-cycles as well. There are six such axes around which the icosahedron can rotate around. The third, and final, type of rotation (called a $T3$ rotation) rotates the icosahedron around an axis created by joining the midpoints of opposite edges. This type of rotation rotates the icosahedron 180^0. This rotation leaves two edges fixed, and rotates the remaining twenty-eight in 2-cycles, and likewise rotates all twenty faces and twelve vertices in 2-cycles. There are fifteen such axes the icosahedron can rotate around. Table 3.3 gathers these rotations in a more organized manner and provides us with the edge, vertex, and face cycle notations, while Figure 3.3 shows how these symmetries act on the icosahedron.

As before, we seek to find $P_F(x_1, x_2, ..., x_{20}), P_V(x_1, x_2, ..., x_{12})$ and $P_E(x_1, x_2, ..., x_{30})$. To find P_F we sum Column 2 of Table 3.3, and get $P_F(x_1, x_2, ..., x_{20}) = \frac{1}{60}(x_1^{20} + 15x_2^{10} + 20x_1^2x_3^6 + 24x_5^4)$. We find the number of 2-colorings by substituting two in for each x_i, $P_F(2, 2, ..., 2) = \frac{1}{60}(2^{20}) + 15(2^{10}) + 20(2^8) + 24(2^4)) = \frac{1}{60}(1069440) = 17824$. To find P_E, we sum Column 4 of Table 3.3 to get $P_E(x_1, x_2, ..., x_{30}) = \frac{1}{60}(x_1^{30} + 15x_1^2x_2^{14} + 20x_3^{10} + 24x_5^6)$. To find the number of 2-colorings, we substitute two in for each x_i, $P_E(2, 2, ..., 2) = \frac{1}{60}(2^{30} + 15(2^{16}) + 20(2^{10}) + 24(2^6)) = \frac{1}{60}(1074746880) = 17912448$.

Figure 3.3: Symmetries of the Icosahedron

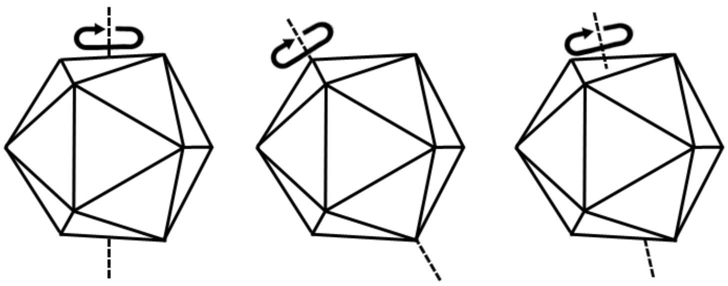


Table 3.3: Face, Vertex, and Edge Cycle Structure Notation of an Icosahedron

Rotation	Face Cycles	Vertex Cycles	Edge Cycles	No. of Axes
Identity	x_1^{20}	x_1^{12}	x_1^{30}	1
120^0T1	$x_1^2x_3^6$	x_3^4	x_3^{10}	10
240^0T1	$x_1^2x_3^6$	x_3^4	x_3^{10}	10
72^0T2	x_5^4	$x_1^2x_5^2$	x_5^6	6
144^0T2	x_5^4	$x_1^2x_5^2$	x_5^6	6
216^0T2	x_5^4	$x_1^2x_5^2$	x_5^6	6
288^0T2	x_5^4	$x_1^2x_5^2$	x_5^6	6
180^0T3	x_2^{10}	x_2^6	$x_1^2x_2^{14}$	15

Now, suppose the king of England wishes to construct a holding device for his most precious stones. He wishes the device to be an icosahedron, where there is a jewel at each vertex, and wants us to find the number of ways he can arrange his stones if he has emeralds, rubies, and diamonds to place at each vertex. We first find $P_V(x_1, x_2, ..., x_{12})$ by summing Column 3 of Table 3.3 and get $P_V(x_1, x_2, ..., x_{12}) = \frac{1}{60}(x_1^{12} + 15x_2^6 + 20x_3^4 + 24x_1^2x_5^2)$. Since we have three types of stones to deal with, we find the number of three colorings of the vertices of the cube by finding $P_V(3, 3, ..., 3) = \frac{1}{60}(3^{12} + 15(3^6) + 20(3^4) + 24(3^4)) = \frac{1}{60}(545940) = 9099$. So, there are 9099 such ways to arrange the precious stones in the device the king envisions.

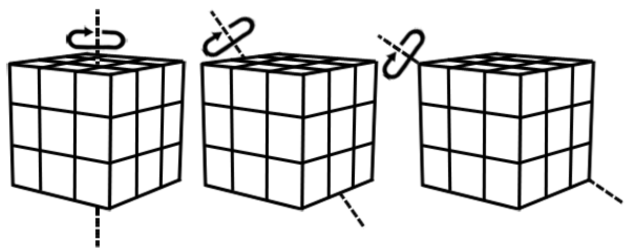
Now, let this be a 2-coloring of the icosahedron, we seek to find the pattern inventory of this 2-coloring (colors are black and white). To achieve this, we find $P_V(b + w, b^2 + w^2, ..., b^{12} + w^{12}) = \frac{1}{60}((b + w)^{12} + 15(b^2 + w^2)^6 + 20(b^3 + w^3)^4 + 24(b + w)^2(b^5 + w^5)^2) = \frac{1}{60}(60b^{12} + 60b^{11}w + 180b^{10}w^2 + 300b^9w^3 + 720b^8w^4 + 840b^7w^5 + 1440b^6w^6 + 840b^5w^7 + 720b^4w^8 + 300b^3w^9 + 180b^2w^{10} + 60bw^{11} + 60w^{12}) = b^{12} + b^{11}w + 3b^{10}w^2 + 5b^9w^3 + 12b^8w^4 + 14b^7w^5 + 24b^6w^6 + 14b^5w^7 + 12b^4w^8 + 5b^3w^9 + 3b^2w^{10} + bw^{11} + w^{12}$.

Application 3.4: $n \times n \times n$ Cube

In Section 1, we examined a $1 \times 1 \times 1$ cube which was free to float in space. Now, we examine a more general case, where each side of the cube can be broken up into n rows or columns. Think of this example as Rubik's Cube where the individual rows and columns are fixed, but the cube itself is free to float in three space. This cube has the same rotations as the $1 \times 1 \times 1$ cube we previously examined. We will consider the rotations around axes that span the midpoints of opposite faces to be $T1$ rotations. These $T1$ rotations rotate the cube $90^0, 180^0$, and 270^0. We also consider the rotations that rotate around axes spanning opposite vertices to be $T2$ rotations. These $T2$ rotations rotate the cube either 120^0 or 240^0. Finally, we consider the rotations that rotate around an axis which joins midpoints of opposite outside edges to be $T3$ rotations. These rotations only rotate the cube 180^0. Additionally, there is the identity rotation which does not rotate the faces, vertices, or edges of the cube. Figure 3.4 shows how these symmetries would act on a $3 \times 3 \times 3$ Rubik's cube.

We know the cube has $f = 6n^2$ faces, but how many unique 2-colorings of those faces are there? Well, we know that the identity rotation will leave all of the faces fixed, thus it will have a cycle structure representation of x_1^f. The $T1$ rotations depend on whether n is even or odd. If n is even, then the $T1$ rotations will not leave any faces fixed, and the $180^0 T1$ rotation will rotate the faces in 2-cycles and the other two will rotate the cube in 4-cycles. Thus, the $180^0 T1$ rotation will have the cycle structure representation $x_2^{\frac{f}{2}}$ when n is even, and the other two $T1$ rotations will have the cycle structure representation $x_4^{\frac{f}{4}}$ when n is even. If n is odd, then the $T1$ rotations will leave two faces fixed, those that the axis passes through, and rotate the rest of the faces as above. Thus the $180^0 T1$ rotation has cycle structure representation $x_1^2 x_2^{\frac{f-2}{2}}$ when n is odd, and the other two $T1$ rotations have cycle structure representation $x_1^2 x_4^{\frac{f-2}{4}}$ when n is odd. Figure 3.4 shows how these symmetries act on the Rubik's cube.

Figure 3.4: Symmetries of a Rubik's Cube



The evenness or oddness of n does not affect how the $T2$ or $T3$ rotations rotate the faces of the cube. We know that the $T2$ rotations move all of the faces of the cube in 3-cycles, so the $T2$ rotations have cycle structure representation $x_3^{\frac{f}{3}}$. Likewise, the $T3$ rotations rotate all of the faces of the cube in 2-cycles, so the $T3$ rotations have cycle structure representation $x_2^{\frac{f}{3}}$. Table 3.4 shows the cycle structure representations for the faces of a $n \times n \times n$ cube.

To find $P_F(x_1, x_2, ..., x_f)$ when n is even, we sum Column 2 of Table 3.4, to get $P_F(x_1, x_2, ..., x_f) = \frac{1}{24}(x_1^f + 9x_2^{\frac{f}{2}} + 8x_3^{\frac{f}{3}} + 6x_4^{\frac{f}{4}})$ for n even. For n odd, we sum Column 3 of Table 3.4 and get $P_F(x_1, x_2, ..., x_f) = \frac{1}{24}(x_1^f + 6x_2^{\frac{f}{2}} + 8x_3^{\frac{f}{3}} + 3x_1^2 x_2^{\frac{f-2}{2}} + 6x_1^2 x_4^{\frac{f-2}{4}})$. To check our work, we set $n = 1$, and find the 2-colorings of these faces. This answer should be the same one we received in section 1. But, if $n = 1$, then $f = 6(1^2) = 6$. As 1 is odd, we use the second formula to receive $P_F(x_1, x_2, ..., x_6) = \frac{1}{24}(x_1^6 + 6x_2^3 + 8x_3^2 + 3x_1^2 x_2^2 + 6x_1^2 x_2)$, and $P_F(2, 2, ..., 2) = \frac{1}{24}(2^6 + 6(2^3) + 8(2^2) + 3(2^4) + 6(2^3)) = \frac{1}{24}(240) = 10$, which is the result we were looking for. Notice, that we also still obtain the pattern inventory, $b^6 + b^5 w + 2b^4 w^2 + 2b^3 w^3 + 2b^2 w^4 + bw^5 + w^6$.

If $n = 2$, then $f = 6(2^2) = 24$, and since n is even, $P_F(2, 2, ..., 2) = \frac{1}{24}(2^{24} + 9(2^{12}) + 8(2^8) + 6(2^6)) = \frac{1}{24}(16816512) = 700688$. Notice that the pattern inventory would be a 24th degree polynomial. To find it we would find $P_F(b + w, b^2 + w^2, ..., b^{24} + w^{24})$.

Table 3.4: Face Cycle Structure Representations of a $n \times n \times n$ Cube

f = $6n^2$

Rotation	Even	Odd	no. of axes
Id	x_1^f	x_1^f	1
$90^0 T1$	$x_4^{\frac{f}{4}}$	$x_4^{\frac{f-2}{4}} x_1^2$	3
$180^0 T1$	$x_2^{\frac{f}{2}}$	$x_2^{\frac{f-2}{2}} x_1^2$	3
$270^0 T1$	$x_4^{\frac{f}{4}}$	$x_4^{\frac{f-2}{4}} x_1^2$	3
$120^0 T2$	$x_3^{\frac{f}{3}}$	$x_3^{\frac{f}{3}}$	4
$240^0 T2$	$x_3^{\frac{f}{3}}$	$x_3^{\frac{f}{3}}$	4
$180^0 T3$	$x_2^{\frac{f}{2}}$	$x_2^{\frac{f}{2}}$	6

Consider now the 2-colorings of the vertices of such a cube. We know that the number of vertices, v, is equal to $6n^2 + 2$. Again, the cycle structure of the $T1$ rotations is affected by whether n is even or odd. If n is even, the axis the cube is rotated around passes through two vertices (the two vertices in the center of the faces of the cube itself that the axis is created with), keeping them fixed. As above, this results in the cycle structure notation of the $180^0 T1$ rotation being $x_1^2 x_2^{\frac{v-2}{2}}$, and the cycle structure of the other two types of $T1$ rotations being $x_1^2 x_4^{\frac{v-2}{4}}$. If n is odd, then the $180^0 T1$ rotation has a cycle structure notation of $x_2^{\frac{v}{2}}$, and the other two types of $T1$ rotations have cycle structure notation of $x_4^{\frac{v}{4}}$.

When considering vertices, the $T3$ type of rotation also depends on whether n is even or odd. Again, if n is even, then the axis the cube rotates around will pass through two vertices (those on the midpoints of the appropriate sides of the cube), and the $T3$ rotations will have cycle structure notation $x_2^{\frac{v-2}{2}}$. If n is odd, then the axis will pass through no vertices, leaving the cycle structure notation $x_2^{\frac{v}{2}}$.

The $T2$ rotations, on the other hand, do not depend on the evenness or oddness of n. In both cases, the axis the cube is rotated around passes through two vertices (the vertices used to create the axis) leaving them fixed, while rotating the other vertices in 3-cycles. The $T3$ rotations will have cycle structure notation of $x_1^2 x_3^{\frac{v-2}{3}}$. The identity rotation has a cycle structure notation of x_1^v. Table 3.5 shows the cycle structure notations for the symmetries of the cube.

We now seek to find P_V for n even and odd. When n is even, we sum Column 2 of Table 3.5 to receive $P_V(x_1, x_2, ..., x_v) = \frac{1}{24}(x_1^v + 9x_1^2 x_2^{\frac{v-2}{2}} + 8x_1^2 x_3^{\frac{v-2}{3}} + 6x_1^2 x_4^{\frac{v-2}{4}})$. In order to receive P_V for n odd, we sum Column 3 of Table 3.5. From this we get $P_V(x_1, x_2, ..., x_v) = \frac{1}{24}(x_1^v + 9x_2^{\frac{v}{2}} + 6x_4^{\frac{v}{4}} + 8x_1^2 x_3^{\frac{v-2}{3}})$.

Table 3.5: Vertex Cycle Structure Notation of a $n \times n \times n$ Cube

v = $6n^2$ +2

Rotation	**Even**	**Odd**	**no. of axes**
Id	x_1^v	x_1^v	1
$90^0 T1$	$x_4^{\frac{v-2}{4}} x_1^2$	$x_4^{\frac{v}{4}}$	3
$180^0 T1$	$x_2^{\frac{v-2}{2}} x_1^2$	$x_2^{\frac{v}{2}}$	3
$270^0 T1$	$x_4^{\frac{v-2}{4}} x_1^2$	$x_4^{\frac{v}{4}}$	3
$120^0 T2$	$x_3^{\frac{v-2}{3}} x_1^2$	$x_3^{\frac{v-2}{3}} x_1^2$	4
$240^0 T2$	$x_3^{\frac{v-2}{3}} x_1^2$	$x_3^{\frac{v-2}{3}} x_1^2$	4
$180^0 T3$	$x_1^2 x_2^{\frac{v-2}{2}}$	$x_2^{\frac{v}{2}}$	6

Now, if $n = 2$, then $v = 6(2^2) + 2 = 26$. So, our $P_V(x_1, x_2, ...x_{26}) = \frac{1}{24}(x_1^{26} + 9x_1^2 x_2^{12} + 8x_1^2 x_3^8 + 6x_1^2 x_4^6)$ since n is even. Thus, $P_V(2, 2, ..., 2) = \frac{1}{24}(2^{26} + 9(2^{14}) + 8(2^{10}) + 6(2^8)) = \frac{1}{24}(67266048) = 2802752$ for two colors with $n = 2$. If $n = 3$, then $v = 6(3^2) + 2 = 56$. So, our $P_V(x_1, x_2, ..., x_{56}) = \frac{1}{24}(x_1^{56} + 9x_2^{28} + 6x_4^{14} + 8x_1^2 x_3^{18})$ since n is odd. Thus, for a 2-coloring, $P_V(2, 2, ..., 2) = \frac{1}{24}(2^{56} + 9(2^{28}) + 6(2^{14}) + 8(2^{20})) = \frac{1}{24}(72057596462333952) = 3002399852597248$ for $n = 3$. The pattern inventory would be a 26th degree polynomial when $n = 2$ and a 56th degree polynomial when $n = 3$. We would find these by finding $P_F(b + w, b^2 + w^2, ..., b^{26} + w^{26})$ and $P_F(b + w, b^2 + w^2, ..., b^{56} + w^{56})$ respectively.

Finally, consider the 2-colorings of the edges of such a cube. We know that the number of edges, e is equal to $12n^2$. When we are talking about the edges of a graph, only the $T3$ rotations of the graph depend on the evenness or oddness of n. When n is even, no edges remain fixed when we rotate a $T3$ rotation (the endpoints of the axis end in vertices as opposed to the midpoints of edges), leaving the cycle structure notation of the $T3$ edge rotations $x_2^{\frac{e}{2}}$. However, when n is odd, then two of the edges (the edges whose midpoints would then make up the endpoints of the axis) remain fixed when we rotate the cube with a $T3$ rotation. The cycle structure notation of the $T3$ rotations becomes $x_1^2x_2^{\frac{e-2}{2}}$ when n is odd. The identity rotation has the cycle structure notation x_1^e as all edges are fixed. The 180^0T1 rotations have the cycle structure notation $x_2^{\frac{e}{2}}$ as no edges remain fixed when $T1$ rotations occur, and this rotation moves the edges of the cube in 2-cycles. Similarly, the other $T1$ rotations leave no edges fixed, but rotate the edges of the cube in 4-cycles. These rotations have the cycle structure notation $x_4^{\frac{e}{4}}$. The $T2$ rotations also leave no edges fixed, but rotate the edges of the cube in 3-cycles. The $T2$ rotations have a cycle structure notation of $x_3^{\frac{e}{3}}$. Table 3.6 gives us a clean way of observing these cycle structure notations.

Table 3.6: Edge Cycle Structure Notation of a $n \times n \times n$ Cube

e = $12n^2$

Rotation	Even	Odd	no. of axes
Id	x_1^e	x_1^e	1
90^0T1	$x_4^{\frac{e}{4}}$	$x_4^{\frac{e}{4}}$	3
180^0T1	$x_2^{\frac{e}{2}}$	$x_2^{\frac{e}{2}}$	3
270^0T1	$x_4^{\frac{e}{4}}$	$x_4^{\frac{e}{4}}$	3
120^0T2	$x_3^{\frac{e}{3}}$	$x_3^{\frac{e}{3}}$	4
240^0T2	$x_3^{\frac{e}{3}}$	$x_3^{\frac{e}{3}}$	4
180^0T3	$x_1^2x_2^{\frac{e-2}{2}}$	$x_2^{\frac{e}{2}}$	6

When n is even, we sum Column 2 of Table 3.6, to obtain $P_E(x_1, x_2, ..., x_e) = \frac{1}{24}(x_1^e + 6x_1^2x_2^{\frac{e-2}{2}} + 3x_2^{\frac{e}{2}} + 8x_3^{\frac{e}{3}} + 6x_4^{\frac{e}{4}})$. For example, if $n = 2$, then $e = 12(2^2) = 48$ and $P_E(x_1, x_2, ..., x_{48}) = \frac{1}{24}(x_1^{48} + 6x_1^2x_2^{23} + 3x_2^{24} + 8x_3^{16} + 6x_4^{12})$. So a 2-coloring of a $2 \times 2 \times 2$ cube would have $P_E(2, 2, ..., 2) = \frac{1}{24}(2^{48}+6(2^{25})+3(2^{24})+8(2^{16})+6(2^{12})) = \frac{1}{24}(281475228917760) = 11728134538240$. The pattern inventory would be the 48th degree polynomial found by $P_E(b+w, b^2+w^2, ..., b^{48}+w^{48})$.

If n is odd, we sum Column 3 of Table 3.6, and obtain $P_E(x_1, x_2, ..., x_e) = \frac{1}{24}(x_1^e + 9x_2^{\frac{e}{2}} + 8x_3^{\frac{e}{3}} + 6x_4^{\frac{e}{4}})$. For example, if $n = 3$, then $e = 12(3^2) = 108$ and $P_E(x_1, x_2, ..., x_{108}) =$

$\frac{1}{24}(x_1^{108} + 9x_2^{54} + 8x_3^{36} + 6x_4^{27})$. For a 2-coloring of such a cube, we obtain $P_E(2, 2, ..., 2) = \frac{1}{24}(2^{108} + 9(2^{54}) + 8(2^{36}) + 6(2^{27})$ which, as 2^{108} is very large, is going to be a very large number. The pattern inventory of this coloring would be found by finding the 108th degree polynomial $P_E(b + w, b^2 + w^2, ..., b^{108} + w^{108})$.

4 Acknowledgement

In writing the first section of this thesis, which is an introduction to polyanumeration, the author often consulted [1]. While [1] was also consulted in the construction of the second and third sections, the author was aided in a couple of examples by [2].

References

[1] A. Tucker, Applied Combinatorics, John Wiley and Sons Inc., Fourth Edition, 2002, 343-368.

[2] M. Eisen, Elementary Combinatorial Analysis, Gordon and Breach, 1969, 199-214