Rochester Institute of Technology

Master's Thesis

A Generalization of the Birkhoff-von Neumann Theorem

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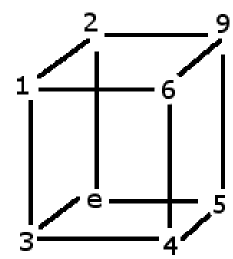
May 18, 2007

In Memory of
My Uncle Scott

Introduction

Hypermatrix: A generalization of the matrix to an $n_1 \times n_2 \times \ldots$ array of numbers.

Example: 3-dimensional hypermatrix (block). This is a $2 \times 2 \times 2$ matrix over $\mathbb{R}$.



Doubly Stochastic Matrix: is a square matrix of non-negative real numbers, each of whose rows and columns sum to 1. (Note: if we can produce a matrix whose rows and columns sum to n, we can still consider it doubly stochastic in nature because we can just divide all the entries by n and get a true doubly stochastic.)

Examples

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1/6 & 5/6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/6 & 0 & 5/6 \\ 5/6 & 0 & 0 & 1/6 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Permutation Matrix: a matrix obtained by permuting the rows of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n. So the number of $n \times n$ permutation matrices is $n!$.

The permutation matrices of order three are:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Birkhoff-von Neumann Theorem: An $n \times n$ matrix over $\mathbb{R}$ is doubly stochastic iff it is a convex linear combination of permutation matrices.

Example:

$$\begin{bmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3/4 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A central idea to keep in mind when trying to prove this theorem, and the generalized version introduced here is the following idea:

Masking: A permutation matrix is said to *mask* a given matrix A if the location of nonzero elements in A corresponds to the locations of the 1's in the permutation matrix.

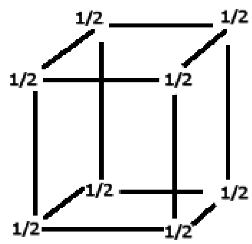
Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ both mask } \begin{bmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix}$$

The reason that this idea is so important is that the permutation matrices can be regarded as a spanning set for doubly stochastic matrices. The ones that will mask a specific doubly stochastic matrix will be found in the convex linear combination and the coefficients can be found by a simple algorithm. This same idea will be used in the proof of the generalized version of the Birkhoff-von Neumann Theorem.

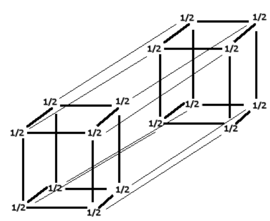
Triply Stochastic Hyper-Matrix (Or Block): a cubical hypermatrix $(n \times n \times n)$ of non-negative real numbers, each of whose rows, columns and depths sum to 1. (Note: if we can produce a matrix whose rows, columns and depths sum to m, we can still consider it triply stochastic in nature because we can just divide all the entries by m and get a true triply stochastic hypermatrix.)

Example



N-tuply Stochastic Matrix: a hypercube matrix of non-negative real numbers each of whose n way arrays sum to 1.

Example Dimension 4 (Tesseract) Quadruply stochastic



Extending Birkhoff:

Birkhoff-von Neumann Theorem for Triply Stochastic Blocks: An $n \times n \times n$ matrix over $\mathbb{R}$ is triply stochastic iff it is a convex combination of permutation blocks.

Permutation Blocks
First we need to create some set of permutation blocks we will call Ω. This will be our new 'spanning' set that will come in handy later on for triply stochastic blocks.

$\Omega = \{\ 0 - 1\ n \times n \times n$ blocks with the property that if (i, j, k) and (i', j', k') are the locations of two different "1-entries" then at most one pair of corresponding indices (i, i'),(j, j'),(k, k') have equal value $\}$

If we are going to create a $1 - 0$ triply stochastic block it must be true that each layer must be made from $1 - 0$ doubly stochastic matrices.

Claim: Our permutation blocks will be constructed with layers that are doubly stochastic $1-0$ permutation matrices.

Proof: We know that the rows, columns and depths sum to 1. So of course fixing a depth and looking at the matrix constructed from the rows and columns will be doubly stochastic because those rows and columns sum to 1. Similarly fixing a row or column will result in the same conclusion. The doubly stochastic matrices here must be $1-0$ permutation matrices, so indeed the permutation blocks of Ω_n are constructed with layers that are $1-0$ permutation matrices.

Notation Label a column as as the integer to which row the 1 appears.

This allows us to represent a permutation matrix as the more commonly used notation for an element of the symmetric group S_n.

The 2×2 *permutation matrices :*

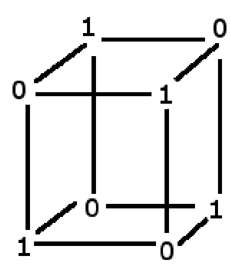
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (12) \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (21)$$

These are the layers of the $2 \times 2 \times 2$ permutation blocks.

Suppose we choose (12) to be our top layer. Then the remaining layer may not have a 1 in the first position of the first column or a 1 in the second position of the second column in order to remain triply stochastic.

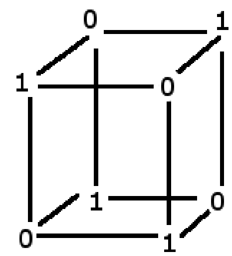
So for $2 \times 2 \times 2$ blocks we get the following:

$$\begin{bmatrix} (12) \\ (21) \end{bmatrix} =$$



Similarly,

$$\begin{bmatrix} (21) \\ (12) \end{bmatrix} =$$



So there are 2 permutation blocks for the $2 \times 2 \times 2$ case.

We will write:

$$\Omega_2 = \left\{ \begin{bmatrix} (12) \\ (21) \end{bmatrix} \begin{bmatrix} (21) \\ (12) \end{bmatrix} \right\}$$

Now lets try to find Ω_3

What are the 3×3 permutation matrices?

$$S_3 = \{(123), (132), (213), (231), (312), (321)\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (123) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = (132) \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (231)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (213) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = (312) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (321)$$

So these will be out layers in our $3 \times 3 \times 3$ permutation blocks.

Suppose (123) is our first layer. Using out previous rule for determining the remaining layers we get the following possibilities:

$$\begin{bmatrix} (123) \\ (231) \\ (312) \end{bmatrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{bmatrix} (123) \\ (312) \\ (231) \end{bmatrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

And with our remaining possibilities we get the following:

$$\Omega_3 = \left\{ \begin{bmatrix} (123) \\ (231) \\ (312) \end{bmatrix} \begin{bmatrix} (123) \\ (312) \\ (231) \end{bmatrix} \begin{bmatrix} (132) \\ (213) \\ (321) \end{bmatrix} \begin{bmatrix} (132) \\ (321) \\ (213) \end{bmatrix} \begin{bmatrix} (213) \\ (321) \\ (132) \end{bmatrix} \begin{bmatrix} (213) \\ (132) \\ (321) \end{bmatrix} \right.$$

$$\left. \begin{bmatrix} (312) \\ (123) \\ (231) \end{bmatrix} \begin{bmatrix} (312) \\ (231) \\ (123) \end{bmatrix} \begin{bmatrix} (213) \\ (321) \\ (132) \end{bmatrix} \begin{bmatrix} (213) \\ (132) \\ (321) \end{bmatrix} \begin{bmatrix} (321) \\ (132) \\ (213) \end{bmatrix} \begin{bmatrix} (321) \\ (213) \\ (132) \end{bmatrix} \right\}$$

Hence, there are 12 permutation blocks for the $3 \times 3 \times 3$ case.

What is very nice about the structure of these permutation blocks is its relation to Euler's Latin squares. In fact the number of elements in Ω_n is the number of Latin squares on n objects.

by definition a **Latin square** consists of n sets of the numbers 1 to n arranged in such a way that no orthogonal (row or column) contains the same number twice.

Counting the number of permutation matrices in higher dimensions involves counting the number of Latin cubes, tesseracts, hypercubes, etc.

Now we have a nice structure for a set of what we might call permutation blocks for 3 dimensional hypermatrices. The next question is whether or not we can represent any triply stochastic block as a convex linear combination of elements in Ω_n (This will generalize the Birkhoff-von Neumann Theorem to the 3-dimensional case).

A $3 \times 3 \times 3$ Triply Stochastic Matrix Represented as a Convex Linear Combination of Elements in Ω_3

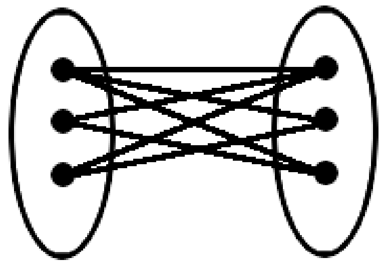
Suppose we start with the following triply stochastic matrix:

$$A = \begin{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{5}{6} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \text{Level 1} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{5}{6} \end{bmatrix} = \text{Level 2} \\ \begin{bmatrix} \frac{1}{6} & 0 & \frac{5}{6} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} = \text{Level 3} \end{bmatrix} \qquad B = \begin{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{bmatrix}$$

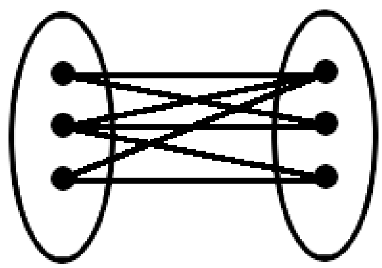
Suppose we also create a $1-0$ matrix block called B (Shown Above) built from the following rule:

$$b_{ijk} = \begin{cases} 0 & \text{if } a_{ijk} = 0 \\ 1 & \text{otherwise} \end{cases}$$

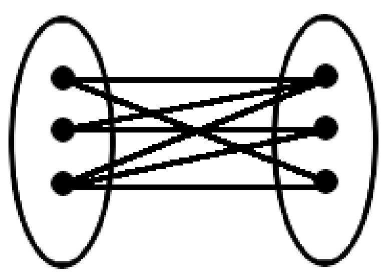
Now build a weighted bipartite graph for each level. The weights are coefficient associated with $a_{row,col,level}$.



Level 1



Level 2



Level 3

For simplicity this was broken up into 3 bipartite graphs but what structure we actually have is a tripartite graph with partied sets: row, col, level (or depth), where each element of out matrix A is a triangle in this tripartite graph.

What we want to do now is find all of the available perfect matchings for each of the bipartite graphs. Each will produce for us a matrix that has nonzero elements in the same positions as a 3×3 permutation matrix. For simplicity I will use the matrix B so the weights will all be 1. This way we actually get $1 - 0$ matrices that are permutation matrices, and then later we will use the weights from our original bipartite graphs.

Layer 1

Perfect Matchings	Associated Permutation Matrix
$\{(1,1),(2,3),(3,2)\}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = (132)$
$\{(1,3),(2,1),(3,2)\}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (231)$
$\{(1,2),(2,3),(3,1)\}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = (312)$

Layer 2

Perfect Matchings	Associated Permutation Matrix
$\{(1,1),(2,2),(3,3)\}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (123)$
$\{(1,2),(2,1),(3,3)\}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (213)$
$\{(1,2),(2,3),(3,1)\}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = (312)$

Layer 3

Perfect Matchings	Associated Permutation Matrix
$\{(1,1),(2,2),(3,3)\}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (123)$
$\{(1,3),(2,1),(3,2)\}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (231)$
$\{(1,2),(2,3),(3,1)\}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (321)$

Now take all of the associated permutation matrices and construct all possible latin squares:

$$\sigma = \left\{ \begin{bmatrix} (132) \\ (213) \\ (321) \end{bmatrix} \begin{bmatrix} (231) \\ (312) \\ (123) \end{bmatrix} \begin{bmatrix} (312) \\ (123) \\ (231) \end{bmatrix} \right\}$$

Now the nice result we were hoping for falls into place where $\sigma \subset \Omega_3$, so we have our permutation blocks that will build the original matrix A in a convex linear combination:

$$A = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

So now we need to find the $\alpha_i' s$.

Algorithm for finding $\alpha_i' s$

Look at all the elements of A that σ_1 has a 1 in the associated position. Choose the minimum entry from those values in A. This will be α_1.

Now subtract $\alpha_1\sigma_1$ off of A.

$$\hat{A} = A - \alpha_1\sigma_1$$

Similarly with $\hat{A}$ look at all the elements that σ_2 has a 1 in the associated position. Choose the minimum entry from those values in $\hat{A}$. This will be α_2. Now subtract $\alpha_2\sigma_2$ from $\hat{A}$

$$\hat{\hat{A}} = \hat{A} - \alpha_2\sigma_2$$

$\hat{\hat{A}}$ will now be some constant multiple of σ_3 and that constant will be α_3.

STEP 1:

$$\alpha_1 = \tfrac{1}{3}$$

$$\hat{A} = A - \tfrac{1}{3}\sigma_1 = A - \tfrac{1}{3}\begin{bmatrix} (132) \\ (213) \\ (321) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \end{bmatrix}$$

STEP 2:

$$\alpha_2 = \tfrac{1}{6}$$

$$\hat{\hat{A}} = \hat{A} - \tfrac{1}{6}\sigma_2 = \hat{A} - \tfrac{1}{6}\begin{bmatrix} (231) \\ (312) \\ (123) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \end{bmatrix}$$

STEP 3:

$$\hat{\hat{A}} = \tfrac{1}{2}\begin{bmatrix} (312) \\ (123) \\ (231) \end{bmatrix}$$
$$\alpha_3 = \tfrac{1}{2}$$

Hence the $\alpha_i' s$ are:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$$

and the convex combination of elements from Ω_3:

$$\begin{aligned} A &= \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 \\ &= \tfrac{1}{3}\sigma_1 + \tfrac{1}{6}\sigma_2 + \tfrac{1}{2}\sigma_3 \\ &= \tfrac{1}{3}\begin{bmatrix} (132) \\ (213) \\ (321) \end{bmatrix} + \tfrac{1}{6}\begin{bmatrix} (231) \\ (312) \\ (123) \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} (312) \\ (123) \\ (231) \end{bmatrix} \end{aligned}$$

The Proof

Theorem 1: A convex linear combination of triply stochastic blocks is triply stochastic.

Proof:
Suppose $\{A_i\}_{i=1}^m$ is a collection of $n \times n \times n$ triply stochastic blocks, and suppose $\{\lambda_i\}_{i=1}^m$ is a collection of scalars satisfying

$$\sum_{i=1}^{m} \lambda_i = 1$$

and $\lambda_i \geq 0$ for each $i = 1, \ldots, m$. We claim that

$$A = \sum_{i=1}^{m} \lambda_i A_i$$

is triply stochastic.

Take any $i \in \{1, \ldots, m\}$. Since A_i is triply stochastic, each of its rows, columns and depths sum to 1. Thus each of the rows, columns and depths of $\lambda_i A_i$ sum to λ_i.

By the definition of element-wise summation, given matrices $N = M_1 + M_2$, the sum of the entries in the ith column of N is clearly the sum of the sums of entries in the ith columns of M_1 and M_2 respectively. A similar result holds for the jth row and the kth depth.

Hence the sum of the entries in the ith column of A is the sum of the sums of entries of the ith columns of $\lambda_q A_q$ for each i, that is,

$$\sum_{q=1}^{m} \lambda_i = 1$$

The sum of the entries of the jth row and kth depth of A is the same. Hence, A is triply stochastic. This proof uses the same techniques found in [Bir] .

Definition: Let B be an $n \times n \times n$ block, Then define:

$B\left(C_i\right)$= The $n \times n$ matrix obtained by fixing the column of B to be the ith column.

Similarly define $B\left(R_i\right)$ and $B\left(D_i\right)$.

Lemma 1: If B is a triply stochastic block, then for every $1 \leq i \leq n$ $B\left(R_i\right)$, $B\left(C_i\right)$ and $B\left(D_i\right)$ are doubly stochastic matrices.

Proof: B is triply stochastic so every row, column, and depth sums to 1. Hence, fixing the row to obtain $B\left(R_i\right)$ does not change the fact that the columns and depths sum to 1. Therefore we know $B\left(R_i\right)$ is indeed doubly stochastic by definition. Similarly for $B\left(C_i\right)$ and $B\left(D_i\right)$ the same is true.

Lemma 2: Every doubly stochastic matrix gives rise to a bipartite graph satisfying the standard Hall condition.

Proof: Suppose B is doubly stochastic. Define a weighted graph $G = (V, E)$ with vertex set $V = \{r_1, \ldots, r_n, c_1, \ldots, c_n\}$, edge set E, where $\omega\left(e_{ij}\right) = B_{ij}$.

Clearly G is a bipartite graph, with partitions $R = \{r_1, \ldots, r_n\}$ and $C = \{c_1, \ldots, c_n\}$, since the only edges in E are between r_i and c_j for some $i, j \in \{1, \ldots, n\}$. Furthermore since $B_{ij} \geq 0$, then $\omega\left(e\right) > 0$ for every $e \in E$.

For any $A \subset V$ define $N(A)$, the neighborhood of A, to be the set of vertices $u \in V$ such that there is some $v \in A$ such that (u, v).

We claim that, for any $v \in V$,

$$\sum_{u \in N(\{v\})} \omega(u, v) = 1$$

Take any $v \in V$; either $v \in R$ or $v \in C$. Since G is bipartite, $v \in R$ implies $N(\{v\}) \subset C$, and $v \in C$ implies $N(\{v\}) \subset R$. Now,

$$v = r_1 \Rightarrow \sum_{u \in N(\{r_i\})} \omega(r_i, v) = \sum_{\substack{j=1 \\ e_{ij} \in E}}^{n} \omega(r_i, c_j) = \sum_{\substack{j=1 \\ B_{ij} \neq 0}}^{n} B_{ij} = \sum_{j=1}^{n} B_{ij} = 1$$

$$v = c_1 \Rightarrow \sum_{u \in N(\{c_i\})} \omega(c_i, v) = \sum_{\substack{i=1 \\ e_{ij} \in E}}^{n} \omega(r_i, c_j) = \sum_{\substack{i=1 \\ B_{ij} \neq 0}}^{n} B_{ij} = \sum_{i=1}^{n} B_{ij} = 1$$

Since B is doubly stochastic. Now, take any $A \subset R$. We have

$$\sum_{\substack{v \in A \\ w \in N(A)}} \omega(v, w) = \sum_{v \in A} \sum_{w \in N(B)} \omega(v, w) = \sum_{v \in A} 1 = \|A\|$$

Let $B = N(A)$. But then clearly $A \subset N(B)$, by definition of neighborhood. So

$$\|N(A)\| = \|B\| = \sum_{\substack{v \in B \\ w \in N(B)}} \omega(v, w) \geq \sum_{\substack{v \in B \\ w \in A}} \omega(v, w) = \sum_{\substack{w \in A \\ v \in N(A)}} \omega(v, w) = \|A\|$$

So $\|N(A)\| \geq \|A\|$.[Bir]

Lemma 3 (Corollary to Lemma 2): Each doubly stochastic matrix M is masked by a permutation matrix of the same size.

Proof: Lemma 2 and the graph-theoretic version of Halls marriage theorem shows there is a perfect matching for the bipartite graph arising from the doubly stochastic matrix. That perfect matching corresponds to a permutation matrix whose nonzero entries corresponding to nonzero entries of M.

Synchronized Pair of Bipartite Graphs with Weighted Edges: Consider the set of ordered triples $\epsilon = \{(i,j,k) \mid 1 \leq i,j,k \leq n\}$. Then define the following subsets:

$$R_{\hat{i}}C_{\hat{j}} = \{(\hat{i},\hat{j},k) \mid 1 \leq k \leq n\}$$
$$C_{\hat{j}}D_{\hat{k}} = \{(i,\hat{j},\hat{k}) \mid 1 \leq i \leq n\}$$
$$R_{\hat{i}}C_{\hat{k}} = \{(\hat{i},j,\hat{k}) \mid 1 \leq j \leq n\}$$

Let $\mathbf{L}$ be the left bipartite graph with disjoint sets $\{R_iC_j \mid 1 \leq i,j \leq n\}$ and $\{R_iD_k \mid 1 \leq i,k \leq n\}$.

Similarly,Let $\mathbf{D}$ be the right bipartite graph with disjoint sets $\{R_iC_j \mid 1 \leq i,j \leq n\}$ and $\{C_jD_k \mid 1 \leq j,k \leq n\}$.

The bipartite graphs are said to synchronized if they only have **allowed edges**; which are:

(For $\mathbf{L}$) There is an edge connecting R_iC_j $R_{i'}D_k$ iff $i = i'$.
(For $\mathbf{D}$) There is an edge connecting R_iC_j $C_{j'}D_k$ iff $j = j'$.

Synchronized Matching: A pair of synchronized bipartite graphs $\mathbf{L}$ and $\mathbf{D}$ is said to have a synchronized matching if there is a subset of edges such that:

1. $\mathbf{L}$ and $\mathbf{D}$ restricted to their subset of edges is still a synchronized pair of graphs.
2. The set of edges induces a perfect matching in both $\mathbf{L}$ and $\mathbf{D}$.

Synchronized Pair of Bipartite Graphs with Weighted Edges Generated by an $n \times n \times n$ **block B**: Suppose $\omega(i,j,k) = B(i,j,k)$. Let $\mathbf{L}$ have vertex sets:

$$\mathbf{L_L} = \{R_iC_j \mid 1 \leq i,j \leq n\}$$
$$\mathbf{L_R} = \{R_iD_k \mid 1 \leq i,k \leq n\}$$

and weighted edge set:

$$E(\mathbf{L}) = \{R_iD_k \overset{\omega(i,j,k)}{\longleftrightarrow} R_iC_j \text{ iff } B(i,j,k) \neq 0\}$$

Similarly, $\mathbf{D}$ has vertex sets:

$$\mathbf{D_L} = \{C_j D_k \mid 1 \le j, k \le n\}$$
$$\mathbf{D_R} = \mathbf{L_R}$$

and weighted edge set:

$$E(\mathbf{D}) = \{R_i D_k \overset{\omega(i,j,k)}{\longleftrightarrow} C_j D_k \text{ iff } B(i,j,k) \neq 0\}$$

Observe that L has an edge precisely when $\omega(i,j,k) \neq 0$ and the same is true of $\mathbf{D}$; so the set of edges satisfies the synchronized property. Also observe that there are no edges in $E(\mathbf{D})$ where $R_i D_k \longleftrightarrow C_j D_{k'}$ for $k = k'$ and similarly for $\mathbf{L}$. So all edges are allowed edges.

Claim 1: If B is a triply stochastic $n \times n \times n$ block, then it induces a pair of synchronized bipartite graphs $\mathbf{L}$ and $\mathbf{D}$.

Proof: In the construction above let there be an edge $R_i C_j \longleftrightarrow R_i D_k$ iff $B(i,j,k) \neq 0$. Then by Lemmas 1 and 2 this gives rise to a synchronized pair of bipartite graphs $\mathbf{L}$ and $\mathbf{D}$.

Pruning:
1. Find a vertex on the left side with the smallest weight and degree greater than or equal to two.

2. On the right find inside the rectangle containing the 'twin' of the selected edge from the left side, a perfect matching containing that 'twin' edge you found on the right and which does not contain any 'isolated edges'. This works because the Birkhoff von Neumann Theorem applies on that perticular rectangle of the right graph since it represents a doubly stochastic matrix.

3. Subtract the smallest among the weights of the edges in the perfect matching from all the edges in the perfect matching found in step 2. If a weight after subtraction becomes zero, erase the edge altogether.

4. On the left side find the twin edges of all the edges affected in step 3 and also subtract from them the value subtracted in step 3. Erase the twins of any edge erased on the right side.

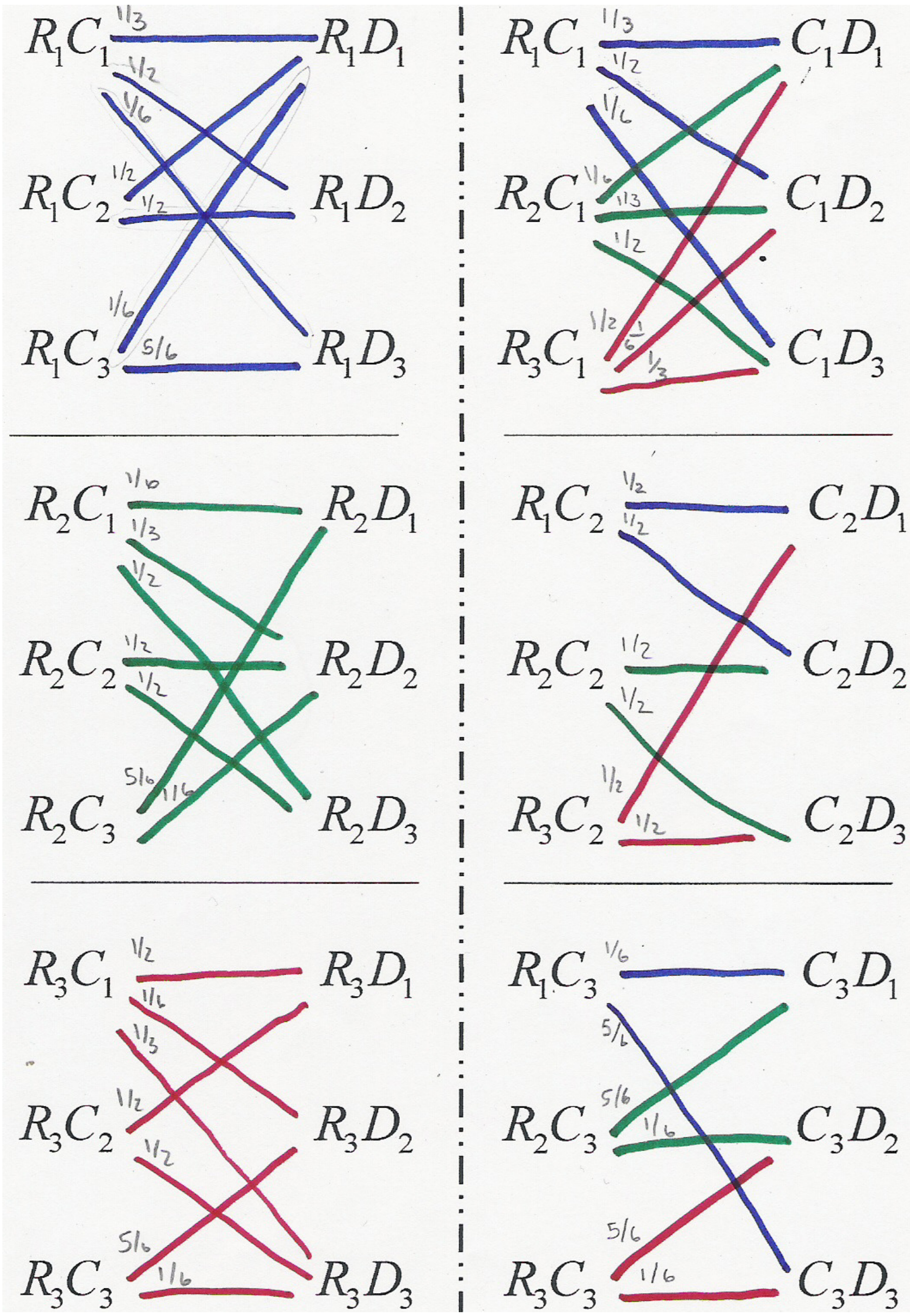
Remarks:

Pruning can be carried out until every edge on the left and right side is an isolated edge. At that point we will have a perfect matching in the left and right bipartite graphs made of pairs of 'twin' edges.

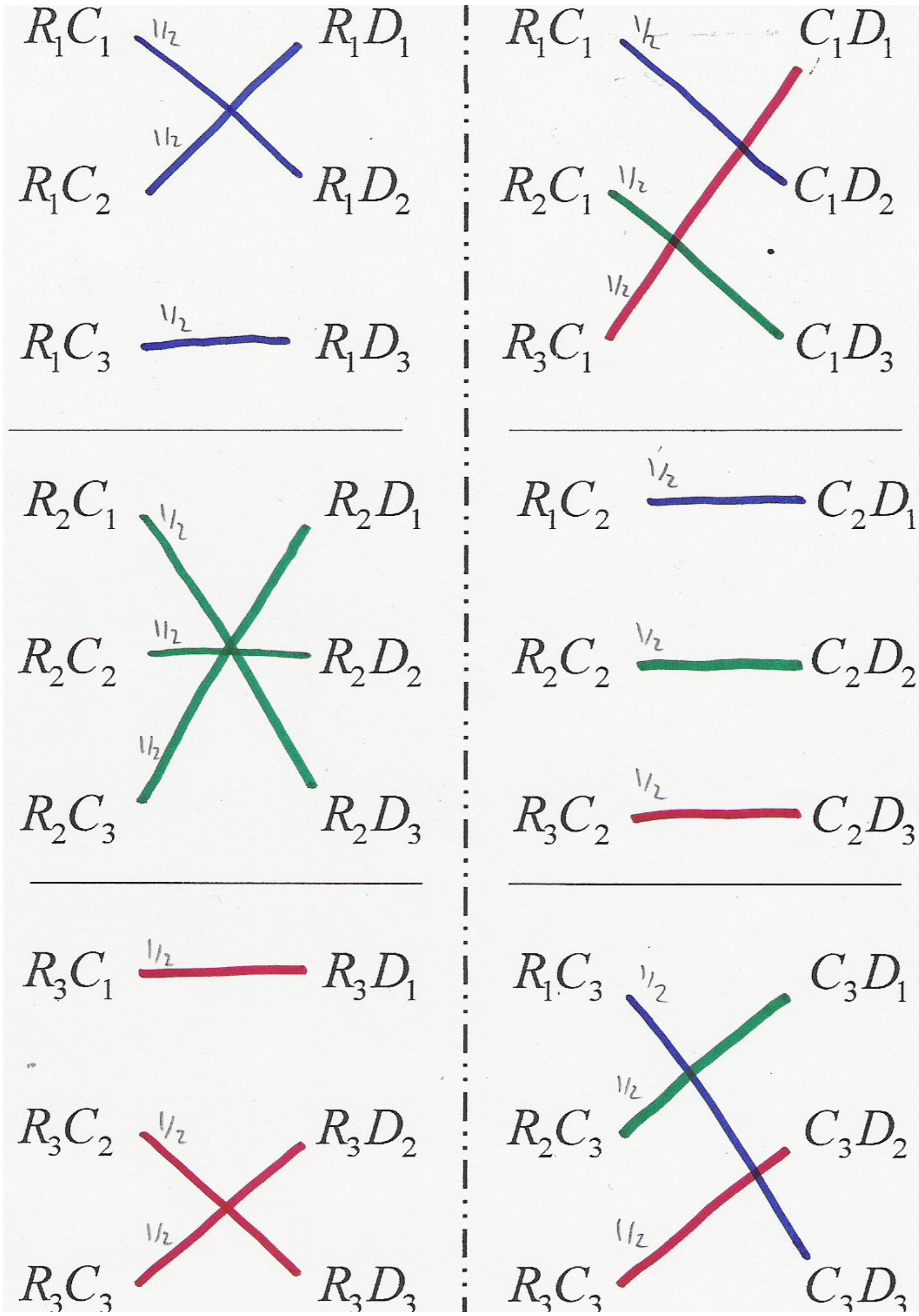
The pair of bipartite graphs correspond to a $Q \in \Omega_n$, with a '1 entry' in position (i,j,k) iff there is an edge $R_i C_j \longleftrightarrow R_i D_k$ (or its twin) in the perfect matching found above.

Now we will run through an example of pruning using the triply stochastic block A on page 8.

Before Pruning



After Pruning



Theorem 2: For every triply stochastic B, we have
$B = \alpha_0 Q_0 + \alpha_1 Q_1 + \ldots + \alpha_{k-1} Q_{k-1}$
where $Q_s \in \Omega_n$ for $1 \leq s \leq k-1$ and $\alpha_0 + \alpha_1 + \ldots + \alpha_{k-1} = 1$

Proof: Suppose $B^{(0)} = B$ and let $B^{(s+1)} = B^{(s)} - \alpha_s Q_s$ where $Q_s \in \Omega_n$ is the block obtained from pruning $B^{(s)}$, and α_s is the smallest weight found in an entry of B corresponding to a '1' entry in Q_s.

By this recursive process of pruning, every $B^{(s+1)}$ is either a zero block or has at least one less entry than $B^{(s)}$. Therefore in a finite number of steps we will reach:

$$0 = B^{(k)} = B - \alpha_1 Q_1 - \alpha_2 Q_2 - \ldots - \alpha_{k-1} Q_{k-1}$$

Hence,

$$B = \alpha_1 Q_1 + \alpha_2 Q_2 + \ldots + \alpha_{k-1} Q_{k-1}$$

with $\alpha_1 + \alpha_2 + \ldots + \alpha_{k-1} = 1$ because block B has row, column and depths that all sum to 1.

Combining theorems 1 and 2 completes the proof for the generalization of the Birkhoff-von Neumann theorem to triply stochastic blocks.

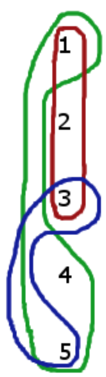
Applications and Open Questions for the Generalization of the Birkhoff-von Neumann Theorem

Systems of Distinct Representatives (SDRs)

Suppose $S = \{S_1, S_2, ..., S_n\}$ is a collection of finite sets. A *system of distinct representatives*, or **SDR** (sometimes also called a transversal), of S is a set of elements $\{x_1, x_2, ..., x_n\}$ such that:

$$x_i \in S_i \text{ for } i = 1, 2, ..., n$$
$$x_i \neq x_j \text{ for } i \neq j$$

Example:



$S = \{S_1, S_2, S_3\}$

$S_1 = \{1, 2, 3\}$

$S_2 = \{1, 4, 5\}$

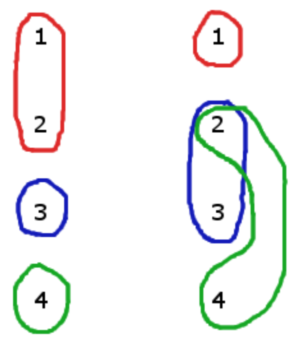
$S_3 = \{3, 5\}$

an SDR: $\{x_1 = 3, x_2 = 4, x_3 = 5\}$

SDR of Two Partitions: Suppose now that $S = \{S_1, S_2, ..., S_n\}$ and $S' = \{S'_1, S'_2, ..., S'_n\}$ are collections of finite sets (We will call S and S' two partitions of our set of elements). An SDR of S and S' is a set of elements $\{x_1, x_2, ..., x_n\}$ such that:

$$x_i \in S_i \text{ and } x_i \in S'_i \text{ for } i = 1, 2, ..., n$$
$$x_i \neq x_j \text{ for } i \neq j$$

Example:



$$S = \{S_1, S_2, S_3\}$$
$$S' = \{S_1, S_2, S_3\}$$

$$S_1 = \{1, 2\} \qquad S'_1 = \{1\}$$
$$S_2 = \{3\} \qquad S'_2 = \{2, 3\}$$
$$S_3 = \{4\} \qquad S'_3 = \{2, 4\}$$

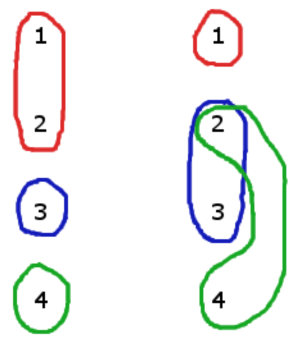
SDR: $\{x_1 = 1, x_2 = 3, x_3 = 4\}$

Suppose we create a Matrix $A = [a_{ij}]$ with the following conditions:

$$a_{ij} = \begin{cases} m & \text{if } S_i \text{ and } S'_j \text{ share } m \text{ elements} \\ 0 & \text{otherwise} \end{cases}$$

We will call this matrix the **Partition Matrix**. (for two partitions)

If we look at the previous example:



$$S = \{S_1, S_2, S_3\}$$
$$S' = \{S_1, S_2, S_3\}$$

$$S_1 = \{1, 2\} \qquad S'_1 = \{1\}$$
$$S_2 = \{3\} \qquad S'_2 = \{2, 3\}$$
$$S_3 = \{4\} \qquad S'_3 = \{2, 4\}$$

Then the partition matrix for these two partitions is the following:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Permanent: The permanent is an analog of the determinant where all the signs in the expansion by the minors are taken as positive.

$$\text{det}(A) = \sum_{\sigma \in S_n} \text{sgn}\,(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

The sum here extends over all elements of the symmetric group S_n.

Example:

$$\text{per} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad + bc$$

The permanent of our defined partition matrix is equal to the number of SDRs for those two partitions.

$$\text{per(A)} = \#\text{ of SDRs}$$

If we look at the partition matrix obtained in the previous example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We find that the permanent will simply just be 1: $Per(A) = 1$. So there is only 1 possible SDR and it was mentioned previously:

$$\textbf{SDR}: \{x_1 = 1, x_2 = 3, x_3 = 4\}$$

This can be seen intuitively because the sets S_1', S_2 and S_3 are singleton sets, and hence must be represented by their only elements, which gives us our only SDR.

Uses for the Permanent: Describes the number of perfect matchings in a bipartite graph. A perfect matching is actually the graph theoretic way of interpreting an SDR essentially.

Example: Let G be a bipartite graph with vertices

$$V(G) = \{A_1, \ldots, A_n, B_1, \ldots, B_n\}$$

where $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ are the two partite sets.

If we represent G as a $n \times n$ matrix $A = a_{ij}$ where:

$$a_{ij} = \begin{cases} 1 & \text{if } \exists \text{ an edge from } A_i \text{ to } B_j \\ 0 & \text{otherwise} \end{cases}$$

The permanent is more difficult to compute. The determinant can be computed in polynomial time by Gaussian Elimination, while the permanent cannot. Computing the permanent of a $0-1$ matrix is actually $\#$P complete.

So now an interesting question we can ask is the following: What conditions do we need to put on two partitions to be guaranteed a doubly stochastic matrix?

Since we have n sets for each partition:

$$S = \{S_1, S_2, \dots, S_n\}$$
$$S' = \{S'_1, S'_2, \dots, S'_n\}$$

and k elements: $\{1, 2, \dots, k\}$

Let's make a function $f : n \times n \longrightarrow \{1, 2, \dots, k\}$

Where $f : (i, j) \longmapsto | S_i \cap S'_j |$

For simplicity I will use 3 sets for each partition. So $S = \{S_1, S_2, S_3\}$ and $S' = \{S'_1, S'_2, S'_3\}$.

(i,j)	$f(i,j)$
(1,1)	$\mid S_1 \cap S'_1 \mid$
(1,2)	$\mid S_1 \cap S'_2 \mid$
(1,3)	$\mid S_1 \cap S'_3 \mid$
(2,1)	$\mid S_2 \cap S'_1 \mid$
(2,2)	$\mid S_2 \cap S'_2 \mid$
(2,3)	$\mid S_2 \cap S'_3 \mid$
(3,1)	$\mid S_3 \cap S'_1 \mid$
(3,2)	$\mid S_3 \cap S'_2 \mid$
(3,3)	$\mid S_3 \cap S'_3 \mid$

If we want to end up with a doubly stochastic partition matrix by nature, then the sum of the rows must be equal to some constant value we will call C. Now we get the following nice result:

$$| S_1 | = | S_1 \cap (S'_1 \cup S'_2 \cup S'_3) | = \sum_{j=1}^{3} f(1, j) = constant = C$$

$$| S_2 | = | S_2 \cap (S'_1 \cup S'_2 \cup S'_3) | = \sum_{j=1}^{3} f(2, j) = constant = C$$

$$| S_3 | = | S_3 \cap (S'_1 \cup S'_2 \cup S'_3) | = \sum_{j=1}^{3} f(3, j) = constant = C$$

Since $\mid S_1 \mid + \mid S_2 \mid + \mid S_3 \mid = k$, then we know $3C = k$. Hence, $C = \frac{k}{3}$.

Similarly, the columns must sum to the same constant value:

$$\mid S_1' \mid = \mid S_1' \cap (S_1 \cup S_2 \cup S_3) \mid = \sum_{i=1}^{3} f(i,1) = constant = C$$

$$\mid S_2' \mid = \mid S_2' \cap (S_1 \cup S_2 \cup S_3) \mid = \sum_{i=1}^{3} f(i,2) = constant = C$$

$$\mid S_3' \mid = \mid S_3' \cap (S_1 \cup S_2 \cup S_3) \mid = \sum_{i=1}^{3} f(i,3) = constant = C$$

and $\mid S_1' \mid + \mid S_2' \mid + \mid S_3' \mid = k$. Hence, $C = \frac{k}{3}$.

So in general if we want to end up with a doubly stochastic partition matrix by nature we must have the following conditions on two partitions of k elements:

$\mid S_j \mid = \mid S_j' \mid = \frac{k}{n}$ for j=1,...,n

Example: Suppose $n = 3$ and $k = 9$. By our previous result we must have the following: $| S_j | = | S'_j | = \frac{9}{3} = 3$ for j=1,2,3.

$S = \{S_1, S_2, S_3\}$
$S' = \{S'_1, S'_2, S'_3\}$

$S_1 = \{1, 5, 6\}$
$S_2 = \{2, 3, 4\}$
$S_3 = \{7, 8, 9\}$
$S'_1 = \{1, 4, 5\}$
$S'_2 = \{2, 3, 7\}$
$S'_3 = \{6, 8, 9\}$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Here the rows and columns sum to 3. So to get a true doubly stochastic matrix we just need to divide each entry of A by 3:

$$\frac{1}{3}A = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

SDR of Three Partitions: Suppose now that $S = \{S_1, S_2, ..., S_n\}$, $S' = \{S'_1, S'_2, ..., S'_n\}$ and $S'' = \{S''_1, S''_2, ..., S''_n\}$ are collections of finite sets (We will call S, S' and S'' three partitions of our set of elements). An SDR of S, S' and S'' is a set of elements $\{x_1, x_2, ..., x_n\}$ such that:

$$x_i \in S_i,\ x_i \in S'_i \text{ and } x_i \in S''_i \text{ for } i = 1, 2, ..., n$$
$$x_i \neq x_j \text{ for } i \neq j$$

Suppose we create a hypermatrix $A = [a_{ijk}]$ with the following conditions:

$$a_{ijk} = \begin{cases} m & \text{if } S_i,\ S'_j,\ S''_k \text{ share } m \text{ elements} \\ 0 & \text{otherwise} \end{cases}$$

We will call this hypermatrix the **Partition Hypermatrix**. (for three partitions)

A Similar question we can ask for three partitions: What conditions do we need to put on three partitions to be guaranteed a triply stochastic block?

Since we have n sets for each partition:

$$S = \{S_1, S_2, \dots, S_n\}$$
$$S' = \{S'_1, S'_2, \dots, S'_n\}$$
$$S'' = \{S''_1, S''_2, \dots, S''_n\}$$

and k elements: $\{1, 2, \dots, k\}$

Let's make a function $f : n \times n \times n \longrightarrow \{1, 2, \dots, k\}$

Where $f : (i, j, k) \longmapsto \mid S_i \cap S'_j \cap S''_k \mid$

For simplicity I will use 3 sets for each partition. So $S = \{S_1, S_2, S_3\}$, $S' = \{S'_1, S'_2, S'_3\}$ and $S'' = \{S''_1, S''_2, S''_3\}$.

(i, j, k)	$f(i, j, k)$
(1,1,1)	$\mid S_1 \cap S'_1 \cap S''_1 \mid$
(1,1,2)	$\mid S_1 \cap S'_1 \cap S''_2 \mid$
(1,1,3)	$\mid S_1 \cap S'_1 \cap S''_3 \mid$
(1,2,1)	$\mid S_1 \cap S'_2 \cap S''_1 \mid$
(1,2,2)	$\mid S_1 \cap S'_2 \cap S''_2 \mid$
(1,2,3)	$\mid S_1 \cap S'_2 \cap S''_3 \mid$
(1,3,1)	$\mid S_1 \cap S'_3 \cap S''_1 \mid$
(1,3,2)	$\mid S_1 \cap S'_3 \cap S''_2 \mid$
(1,3,3)	$\mid S_1 \cap S'_3 \cap S''_3 \mid$
(2,1,1)	$\mid S_2 \cap S'_1 \cap S''_1 \mid$
⋮	⋮
(3,3,3)	$\mid S_3 \cap S'_3 \cap S''_3 \mid$

If we want to end up with a triply stochastic partition hypermatrix by nature, then the sum of the depths must be equal to some constant value we will call C. Now we get the following nice result:

$$\mid S_1 \cap S'_1 \mid = \mid S_1 \cap S'_1 \cap (S''_1 \cup S''_2 \cup S''_3) \mid = \sum_{k=1}^{3} f(1, 1, k) = constant = C$$

$$\mid S_1 \cap S'_2 \mid = \mid S_1 \cap S'_2 \cap (S''_1 \cup S''_2 \cup S''_3) \mid = \sum_{k=1}^{3} f(1, 2, k) = constant = C$$

$$| S_1 \cap S_3' |=| S_1 \cap S_3' \cap (S_1'' \cup S_2'' \cup S_3'') |= \sum_{k=1}^{3} f(1,3,k) = constant = C$$

$$| S_2 \cap S_1' |=| S_2 \cap S_1' \cap (S_1'' \cup S_2'' \cup S_3'') |= \sum_{k=1}^{3} f(2,1,k) = constant = C$$

$$\vdots$$

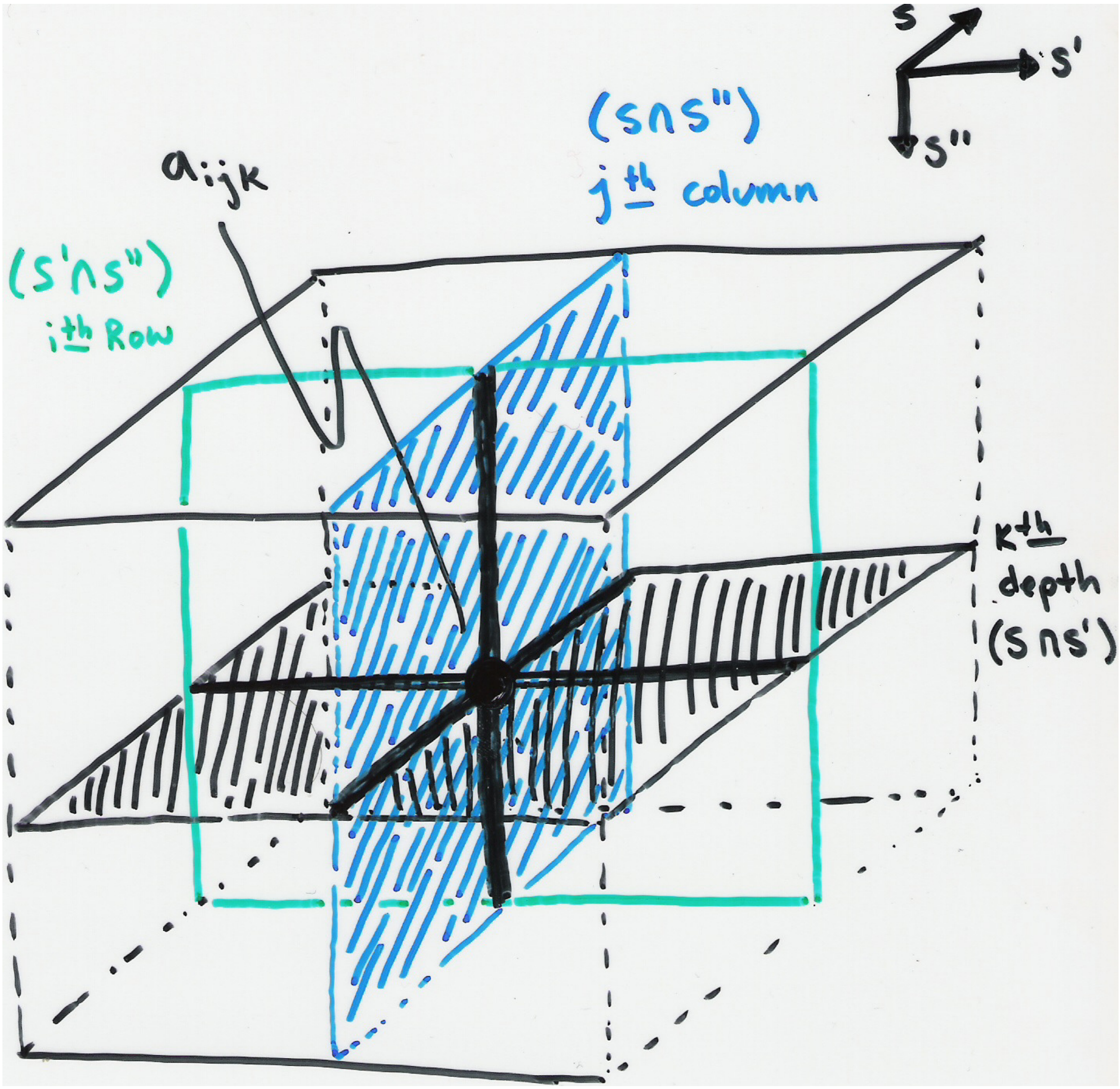
$$| S_3 \cap S_3' |=| S_3 \cap S_3' \cap (S_1'' \cup S_2'' \cup S_3'') |= \sum_{k=1}^{3} f(3,3,k) = constant = C$$

If we look at the partitions corresponding to the row and column sums we would get similar results, and in general we can make the following claim:

Let S, S' and S'' be three different partitions of a set with k elements satisfying the condition that for any $x \in S$, $x' \in X'$ and $x'' \in S''$, we have:

$$| S \cap S' |=| S \cap S'' |=| S' \cap S'' |$$

then the partition hypermatrix produced from these three partitions will be triply stochastic by nature. A nice way to visualize what is going on here is on the following page.


S
S'
S"
(S∩S")
jth column
a_{ijk}
(S'∩S")
ith Row
kth depth
(S∩S')

Hyperpermanent:

3-dimensional matrix (block)

$$\sum_{\sigma,\phi\in S_n}\prod_{i=1}^{n} a_{i,\sigma(i),\phi(i)}$$

This is almost what you would expect. Now the third index is just running over another permutation that is some element in the symmetric group S_n. And even more generally we get the following:

m-dimensional matrix

$$\sum_{\sigma_1,\ldots,\sigma_m\in S_n}\prod_{i=1}^{n} a_{i,\sigma_1(i),\ldots,\sigma_m(i)}$$

Lower Bounds for the Hyper-Permanent of Triply Stochastic Blocks (Hypermatrices)

Van der Waerden's Permanent conjecture:

Let A be an $n \times n$ matrix.
Let $\hat{A}$ be the one where all the entries are equal (ie $\frac{1}{n}$). Its permanent.

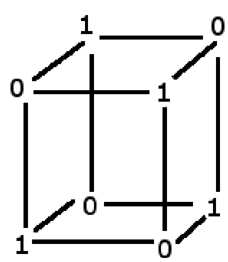
$$\text{per}\hat{A} = n! \left(\frac{1}{n}\right)^n$$

and Van Der Waerden conjectured in 1926 that the smallest value for any doubly stochastic A, is attained only for $A = \hat{A}$.

$$\text{per}A > n! \left(\frac{1}{n}\right)^n (\text{for} A \neq \hat{A})$$

It was finally proven independently by Egorychev and by Falikman in 1979/80.

A nice result does not seem to exist for triply stochastic matrices at first glance:



Has permanent zero.

Property of permanents that determinants don't have

$$\text{perm}(A) \geq \min\{\text{perm}(P_1), \text{perm}(P_2), \ldots, \text{perm}(P_n)\}$$

given

$$A = k_1P_1 + k_2P_2 + \ldots + k_nP_n = \sum_{i=1}^{n} k_iP_i$$

such that

$$k_1 + k_2 + \ldots + k_n = \sum_{i=1}^{n} k_i = 1 \text{ for } k_i \geq 0$$

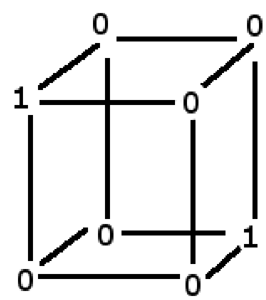
Together with the Birkhoff-von Neumann Theorem for Triply Stochastic Blocks we can say that if we find a lower bound for all of the permutation blocks, then this will also be a lower bound for all triply stochastic blocks of the same dimensions as the permutation blocks. In other words, if all the elements of Ω_n have a lower bound then so does every $n \times n \times n$ triply stochastic block.

So the lower bound for $2 \times 2 \times 2$ Triply Stochastic Blocks is zero. Is this true for higher values of n for $n \times n \times n$ Triply Stochastic Blocks?

Suppose we have the following $3 \times 3 \times 3$ Triply Stochastic Permutation Block:

$$B = \begin{bmatrix} (123) \\ (231) \\ (312) \end{bmatrix}$$

If we can find one diagonal that produces a nonzero multiplication than we know that the overall permanent must be greater than zero. Suppose we choose the upper left value in the block which is a 1. Then if we look at the remaining $2 \times 2 \times 2$ minor we need to find the permanent of that and multiply it by 1. The remaining minor is the following:



And clearly this $2 \times 2 \times 2$ minor has a nonzero permanent. Hence, $Per(B) > 0$. This same result can be seen with the other remaining 11 elements of Ω_3.

The big question remains: What dimensions produce a nonzero lower bound? What is the lower bound? Can we get something as nice as the Van Der Waerden conjecture for triply stochastic blocks, or even n-tuply stochastic hypermatrices?

Latin Squares

There is no known easily-computable formula for the number of $n \times n$ Latin squares with n objects (Remember this is also the number of elements in Ω_n). The most accurate upper and lower bounds known for large n are far apart.

But an interesting relation to the previous questions is the following:

If you look at all the nonzero positions of the $n \times n$ permutation matrices, and match those positions to each element of Ω_n

Take an element of Ω_n and mask each $n \times n$ permutation matrix on top of it. If all the nonzero positions of each permutation matrix are unique, in other words if all n objects are represented, then that element of Ω_n has a nonzero permanent. This is because each permutation matrix represents a possible diagonal in Ω_n when it is represented in its latin square form. So if we end up with some diagonal in the latin square which represents all n objects, then there is a nonzero multiplication present that will produce a nonzero permanent.

Example:

Take out B from before:

$$B = \begin{bmatrix} (123) \\ (231) \\ (312) \end{bmatrix}$$

and look at the permutation matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Masking the permutation matrix on top of B gives us elements 1,3,2. Hence the permanent of B is nonzero, as we showed before.

Similarly, looking at an element of Ω_2:

$$C = \begin{bmatrix} (12) \\ (21) \end{bmatrix} \text{ and permutation matrices: } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We see there will be a zero permanent for C. This approach to solving some of the previous questions might also bring some interesting results.

Birkhoff Polytope

The class of $n \times n$ doubly stochastic matrices is a convex polytope in $\mathbb{R}^N$ (where $N = n^2$) known as the *Birkhoff polytope*, B_n. It's dimension is $(n-1)^2$ since the line sums being equal to 1 imposes $2n - 1$ linear constraints (not $2n$, because if the total of all n columns is n, the same must be true of all n rows.)

Another way wording the Birkhoff-von Neumann Theorem is the following:

The set B_n of doubly stochastic matrices of order n is the convex closure of the set of permutation matrices of the same order, and furthermore that the vertices (extreme points) of B_n are precisely the permutation matrices.

And hence our generalized version would say something like the following:

The set C_n of triply stochastic blocks of order n is the convex closure of Ω_n, and furthermore that the vertices (extreme points) of C_n are precisely the elements in Ω_n.

If we take the two elements of Ω_2 and make them the vertices in a convex polytope then we get a line segment:

If these elements span all triply stochastic $2 \times 2 \times 2$ matrices by a convex linear combination, then any point on the indicated line will represent some $2 \times 2 \times 2$ triply stochastic matrix.

As we go to large Ω_n our convex polytope becomes much more involved.

A big question that some people have been trying to answer is the volume of B_n (which is the regular Birkhoff Polytope made from normal $n \times n$ doubly stochastic matrices). So far the volume up to B_{10} is known. Here we might wonder what kinds of connections does this new polytope have with the previously defined Birkhoff Polytope. The extreme vertices in this new polytope being the latin squares on n objects grows much more rapidly, so is trying to find the volume of these polytopes even a reasonable question? What kinds of convex shapes can we say are allowable for this new Polytope?

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