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What is 3.4 Parts Per Million?

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Abstract. There is only a weak and poorly defensible statistical link between the notion of a so-called 6σ process and the 3.4 parts per million defect rate that is usually associated with it in the Six-Sigma literature. In this note, we comment on the purported link and illustrate two reasonable ways in which a process may be related to a defect rate. First, we consider the average defect rate generated when a process goes out of control due to a $k\sigma$ shift and runs there under the shift is detected on an $\bar{X}/R$ chart. Second, we consider the average defect rate when the process mean drifts in a sinusoidal fashion with a maximum drift of $\pm k\sigma$ from target. In neither case is there a natural connection to a 3.4 pars per million defect rate. We assert that a true 6σ process should be associated with the more natural and more easily explained 0.002 parts per million defect rate.

Keywords: six sigma, defect rate, statistical process control, engineering process control

1. Introduction

Suppose the following quiz was given to a group of Six-Sigma professionals.

> If the quality characteristic being measured has a normal distribution with mean μ and standard deviation σ, then 3.4 parts per million is
>
> (a) the fraction that falls outside of $\mu \pm 3\sigma$
>
> (b) the fraction that falls outside of $\mu \pm 6\sigma$
>
> (c) the fraction that falls outside of $\mu \pm 4.5\sigma$
>
> (d) none of the above.

How many would know the correct answer?

When Motorola was connecting the notion of six sigma to ppm (parts per million) levels, they assumed that a process might shift 1.5σ from its target. They based 3.4 ppm on such an assumption. "Assumption" is used, because the author is unaware of any formal analysis that was done to substantiate this claim. So, the 3.4 ppm is connected, in some way, with a putative 1.5σ shift.

Here is a review of the possible quiz answers. Answer (a) is incorrect. The fraction that falls outside of this range is actually 2,700 ppm, so is definitely not Six-Sigma quality. Answer (b) is also incorrect. The actual fraction here is only 0.002 ppm.

This seems to be the natural ppm number to associate with Six Sigma. Answer (c) is closer to the truth, but is still not correct. The fraction that falls outside of that range is actually 6.8 ppm. The correct answer is (d), "none of the above." The actual value of 3.4 ppm refers to the fraction that would fall *either* above $\mu + 4.5\sigma$ or below $\mu - 4.5\sigma$.

It is ironic that part of the very nature of Six Sigma, precision of operation, is abandoned by this awkward and tenuous connection. Stephens (2001) makes a similar point.

In this note, we consider several simple process scenarios to see what ppm levels would be generated by each of them. For each, we will keep this 1.5σ shift in mind. For each scenario, we begin by assuming that the process otherwise *runs ideally.* By this, we mean it is centered within two-sided specification limits, and that these limits are symmetrical and precisely at $\mu \pm 6\sigma$. That is, the process is on target and 6σ away from each limit. This ideal process would produce the 0.002 ppm defect rate.

The first scenario is the "Motorola Scenario." There, to "achieve" 3.4 ppm, a process could always run either 1.5σ above the target, or 1.5σ below the target. In such a case, the 3.4 ppm defect rate would occur. But this is totally unrealistic for what is supposed to be a (controlled) Six-Sigma process, as we will show in the next section.

2. Statistical Process Control Scenario

In a second scenario, say the process suddenly undergoes, e.g., a 1.5σ shift. This is also not too realistic, but could happen if, say, a machine was set up improperly. Say this happens at the beginning of the week on a process that is run for an eight-hour shift, seven days per week. Also, say the process is being monitored (part of the last step in the define-measure-analyze-improve-control sequence) using a standard $\bar{X}/R$ control chart with a subgroup size of $n = 4$ every two hours. Then in each subgroup, it is straightforward to show that there is a 50% chance of detecting this 1.5σ shift, using the standard "± 3 sigma" control-chart detecting rule. Suppose that once this out-of-control condition is detected that the process is corrected back to target. This means that the process would run at a 3.4 ppm defect rate for 2 hours with a 50% chance, for 4 hours with a 25% chance, for 8 hours with a 12.5% chance, and so on. Then, even if such product were released, the average ppm defect rate for the entire week turns out to be only 0.24 ppm, or 240 ppb. This is because the monitoring process would most likely catch the out-of-control condition very early in the week.

In fact, if we consider different amounts of sudden process shifts in this scenario, we would need to have a sudden process shift of 2.25σ to have a 3.4 ppm for the entire week. This is shown in Figure 1, where the results are plotted on a ppb (parts per billion) scale for better readability. Also, recall that no shift corresponds to 2 ppb and a 1.5σ shift to 240 ppb. Even if a sudden 2.25σ shift were to occur, this

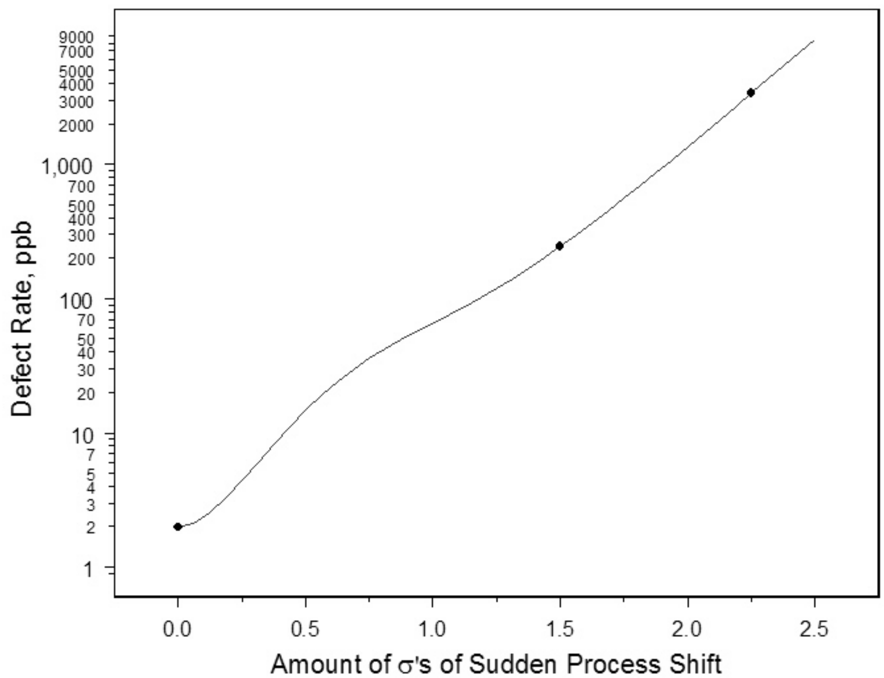

Figure 1: Weekly Defect Rate vs. Amount of Sudden Process Shift at Beginning of the Week.

would shift the process up 4.5 $\sigma_{\bar{X}}$'s on the control chart, would be detected with a 93% chance at the first subgroup, and would very possibly result in some action to prevent the release of the material produced during that short time frame.

Now consider a the more general version of this scenario. Say the process undergoes a $k\sigma$ shift at time 0. At times $t = 1, 2, \ldots$ a subgroup of size n is taken and the process is tested for an out-of-control condition by using the "± 3 sigma" control-chart rule. As soon as the process is detected as being out of control, the process is re-centered on target. The process then stays in control up to time τ. (Only the ratio of t/τ matters, so we have chosen to sample t only at values $1, 2, \ldots$) Consider the average defect rate of such a process betwee 0 and τ. We find:

1. The time to detect the out-of-control condition is a geometric random variable

T with parameter p_k, where

$$p_k = 1 - \Phi\left(3 - k\sqrt{n}\right) + \Phi\left(-3 - k\sqrt{n}\right)$$

and Φ is the standard normal cdf. The first two terms here are associated with the probability of a detection in the correct direction. The last term has little effect on the results.

2. Before the condition is corrected, the defect rate is $d_k = 1 - \Phi(6-k) + \Phi(-6-k)$. After it is corrected, the defect rate is $d_0 = 1 - \Phi(6) + \Phi(-6)$.

3. When the condition is detected (and corrected) at time T, and $T < \tau$, then the average defect rate from time 0 up to time τ is the weighted average $(d_k T + d_0(\tau - T))/\tau$. If the condition is not detected before time τ, then the average defect rate is $d_k \tau/\tau = d_k$.

4. Thus, the overall average defect rate is

$$ADR_k = \sum_{t=1}^{\tau-1} \left[(1-p_k)^{t-1} p_k \left(d_k t + d_0(\tau - t)\right)/\tau\right] + (1-p_k)^{\tau} d_k.$$

(Note that ADR_k is actually a function of k, n, and τ, but we write "ADR_k" because we primarily consider it as a function of k.) Consider what happens for large values of k. From the nature of this process, the process would be corrected with a very high probability in the first time period, so $ADR_k \approx (d_k + d_0(\tau-1))/\tau$. If τ is not too large (relative to $1/d_0$), then $ADR_k \approx 1/\tau$ for large values of k. For example, See Figure 2, in which $n = 4$ and $\tau = 100$ was used.

It is also reasonable that ADR_k should be an increasing function of k (for $k > 0$), and this is visually confirmed for the special case shown in Figure 1. For $k > 0$, it is intuitive and is easy to show that d_k increases in k. If this were the only quantity in ADR_k that varied with k, then in fact ADR_k would clearly increase with k. However, by the same mathematics, p_k increases in k, and this increase by itself tends to produce lower values of ADR_k. So it is the interplay between these two functions of k that will determine the overall behavior of ADR_k. We have not investigated this further.

3. Engineer Process Control Scenario

In a third scenario, consider a process that drifts in a certain pattern. A controller is attached to the process, and this keeps the process on target on an average long-term basis. However, the process mean drifts in a sinusoidal (sine-wave) pattern of $\pm k\sigma$ around the target. See Figure 3 for an example of a pattern of $\pm 1\sigma$. Such controllers are common in industry, and is an example of so-called engineering process control

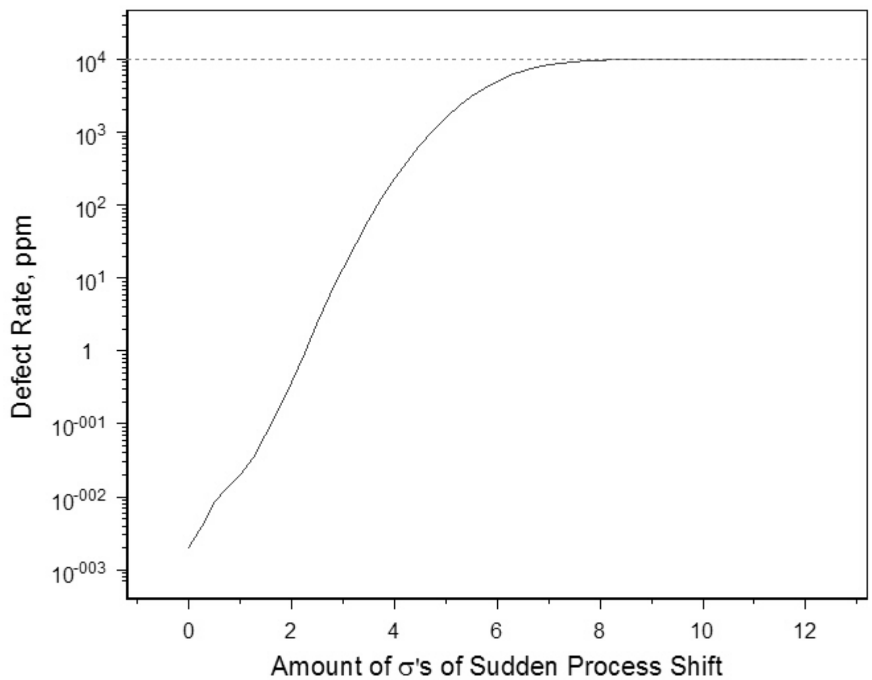

Figure 2: Limiting Case for $\tau = 100$, where Defect Rate Approaches $1/\tau = 10^4$ ppm.

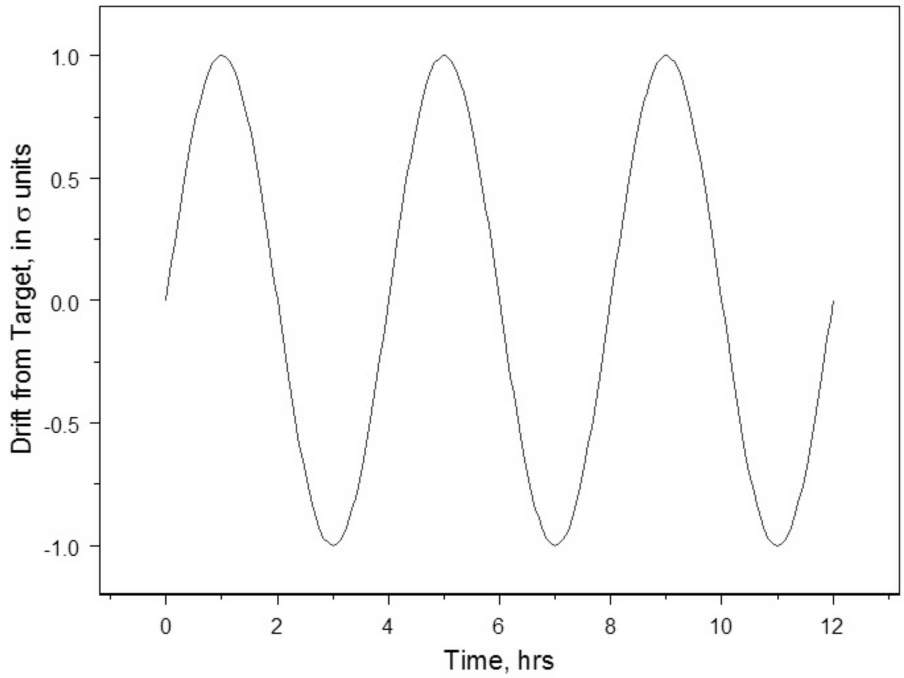

Figure 3: Amount that Process Mean Drifts from Target over Time—Maximum 1σ-Shift Example.

instead of statistical process control. For an understanding of both control methods from a statistical perspective, see Box and Luceño (1997).

Consider how this drift would affect an otherwise ideal process. Because of the ideal process assumptions and the symmetry shown in Figure 3, we only need to consider the process behavior for $\frac{1}{4}$ of the a cycle, say from 0 to $\pi/2$. At time t in this cycle, the defect rate will be

$$d\left(k,t\right)=1-\Phi\left(6-k\sin\left(t\right)\right)+\Phi\left(-6-k\sin\left(t\right)\right).$$

Thus, the average defect rate during this $\frac{1}{4}$ cycle is

$$ADR_k=\frac{\pi}{2}\int_0^{\pi/2}d\left(k,t\right)\ dt.$$

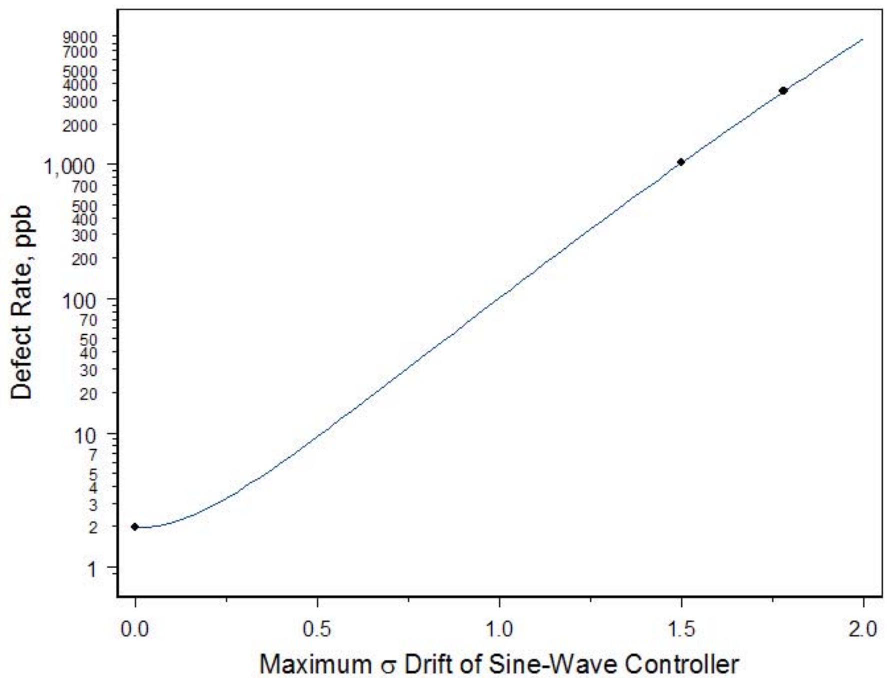

Figure 4: Defect Rate vs. Maximum Amount of Off-Target Drift of Process Mean from Target.

If the maximum process shift that occurs under this scenario is 1.5σ ($k = 1.5$), then the resulting defect rate is 1.02 ppm. (Integral calculations were performed in Splus®.) This defect rate is higher than that due to the sudden process shift scenario of 1.5σ for two reasons. First, there is no "detection" step that would lead to adjusting the process back to the target value. Second, the nature of the sine function means that the process spends more time near the extreme values of the drift than in the center of the process. For example, in Figure 2, the process is more than 0.5σ from the target—$\frac{1}{2}$ of the maximum drift—for ■ of the total process time.

Still, even if this case, it would take a maximum process drift of 1.78σ to produce a 3.4 ppm defect rate. This value, as well as those corresponding to 0σ and 1.5σ, are highlighted in Figure 4. So, in this scenario as well, there is no natural association between 3.4 ppm and the process.

4. Conclusions

Is there a rational place for the phrase “3.4 ppm” in a Six-Sigma world? Based on the implausibility of the original explanation of the supposed 1.5σ shift, and based on some reasonable scenarios, no reasonable connection between the 3.4 ppm level and the 6σ level can be found. In spite of this, this 3.4 ppm level will most likely stay, and will continue to be associated—incorrectly—with a 6σ process.

5. Appendix

This Appendix contains the Splus® code used to create the values for the graphs that appear in this report.

```
# Define function to find E(PPM)
"Eppm"<-function(k=1.5,n=4,tau=28) {
# Assumes ideal process is centered around specs and is
# +/- 6 s from the spec limits.
# Assumes time period is broken up into tau intervals.
# E.g.  this gives tau=28:
# Time period=One week with 8-hour, 7-day work week = 56 hours,
# with 2-hour sampling intervals.
# ks shift occurs at time 0.
# At each interval, starting at 1, subgroup size n is taken ->
# A "p" chance of detecting the ks shift.
# When shift is detected, the process is recentered, and stays
# centered for the rest of the time period.
# Based on this, E(PPM) is calculated for the entire time period.
# P of detecting shift in any subgroup, including wrong direction
p<-1-pnorm(3-sqrt(n)*k) + pnorm(-3-sqrt(n)*k)
ppm.ks<-(10**6)*((1-pnorm(6-k))+pnorm(-6-k)) # ppm if off by ks
ppm.0s<-(10**6)*pnorm(-6)*2 # ppm if off by 0 sigma (6-sig process)
t<-1:tau
# prob of detecting exactly at interval t, for t=1:(tau-1)
p.det<-p*((1-p)**(0:(tau-2)))
p.det[tau]<-1-sum(p.det) # prob of not detecting by time tau-1
# =prob of detecting at time tau or later, which means the process
# is out of control the entire time period
# vector of E(PPM)if detected at interval t, for t=1:tau
ppm.terms<-(t*ppm.ks+(tau-t)*ppm.0s)/tau
crossprod(ppm.terms,p.det) # E(PPM)
}
# Find E(PPM) for kk-sigma shifts shown in next line
kk<-(0:40)/16
```

```
i<-0
eppm<-kk # only to set size of vector eppm
for(k in kk) {
    i<-i+1
    eppm[i]<-Eppm(k)
}
# Print out vectors, find E(PPB), define points to highlight,
# arrange into a data frame
eppm
kk
eppb<-eppm*(10**3)
ids<-0*eppb # used to flag certain point
ids[c(1,25,37)]<-1
eppm.kk<-as.data.frame(cbind(eppm,eppb,kk,ids))
# Make up graph using S-Plus menus...
# Make up sine-wave example
Time<-seq(0,12,by=.05)
Drift<-sin(Time*pi/2)
Sin.labels<-c("-1s","-0.5s","0s","0.5s","1s")
Sin.Curve<-as.data.frame(cbind(Time,Drift))
# Make up graph using menus...
# Find ppm of process with sine-wave drift.
# Need to do numerical integration here.
# kk<-(0:40)/16
kk<-(0:64)/32
i<-0
eppm<-kk # to set size
for(k in kk) {
    i<-i+1
    integrand <- function(x) {
        (10**6)*((1-pnorm(6-k*sin(x)))+pnorm(-6-k*sin(x)))}
    result<-integrate(integrand, lower = 0, upper = pi)
    eppm[i]<-result$integral/pi
}
eppm
kk
eppb<-eppm*(10**3)
ids<-0*eppb # used to flag certain point
ids[c(1,49,58)]<-1
Sin.kk<-as.data.frame(cbind(eppm,eppb,kk,ids))
```

```
# Make up graph using menus...
# test specific values of k, to find k value leading to 3.4 ppm
k<-1.7756
integrand <- function(x) {
    (10**6)*((1-pnorm(6-k*sin(x)))+pnorm(-6-k*sin(x)))}
result<-integrate(integrand, lower = 0, upper = pi)
result$integral/pi
# Look at limiting k case
kk<-(0:48)/4
i<-0
eppm<-kk # to set size
for(k in kk) {
    i<-i+1
    eppm[i]<-Eppm(k,tau=100)
}
eppm
kk
ids<-0*eppm # used to flag certain point
ids[c(1,25,37)]<-1
eppm.limitk<-as.data.frame(cbind(eppm,kk,ids))
# Make up graph using menus...
```

6. Bibliography

References

[1] Box, G., and Luceño, A. (1997), *Statistical Control: By Monitoring and Feedback Adjustment*, Wiley-Interscience.

[2] Stephens, K. (2001), "What is the Future of Six Sigma," Six Sigma Forum Magazine, November, 47-48.