HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

To determine the general solution to homogeneous second order differential equation:
\[ y'' + p(x) y' + q(x) y = 0 \]

Find two linearly independent solutions \( y_1 \) and \( y_2 \) using one of the methods below.

Note that \( y_1 \) and \( y_2 \) are **linearly independent** if there exists an \( x_0 \) such that Wronskian
\[
W(y_1, y_2)(x_0) = \det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\
y'_1(x_0) & y'_2(x_0) \end{bmatrix} = y_1(x_0)y'_2(x_0) - y_2(x_0)y'_1(x_0) \neq 0
\]

The general solution is \( y(x) = C_1 y(x) + C_2 y(x) \) where \( C_1 \) and \( C_2 \) are arbitrary constants.

### METHODS FOR FINDING TWO LINEARLY INDEPENDENT SOLUTIONS

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| **Reduction of order** | Given one non-trivial solution \( f(x) \) to \( y'' + p(x) y' + q(x) y = 0 \) | Either:  
1. Set \( y(x) = v(x) \cdot f(x) \) for some unknown \( v(x) \) and substitute into differential equation.  
2. Now we have a separable equation in \( v' \) and \( v'' \). Use the Integrating Factor Method to get \( v' \) and then integrate to get \( v \).  
3. Substitute \( v \) back into \( y(x) = v(x) \cdot f(x) \) to get the second linearly independent solution.  
Or: 
\[
y(x) = f(x) \cdot \int e^{-\int p(x) dx} \frac{f(x)}{[f(x)]^2} \, dx
\]
where \( y(x) \) is the second linearly independent solution. |
| **Characteristic (Auxiliary) Equation** | \( a y'' + b y' + c y = 0 \) where \( a, b \) and \( c \) are constants | 1. Find solutions \( r_1 \) and \( r_2 \) to the characteristic (auxiliary) equation: \( ar^2 + br + c = 0 \)  
2. The two linearly independent solutions are:  
   a. If \( r_1 \) and \( r_2 \) are two real, distinct roots of characteristic equation:  
      \( y_1 = e^{r_1 x} \) and \( y_2 = e^{r_2 x} \)  
   b. If \( r_1 = r_2 \) then \( y_1 = e^{r_1 x} \) and \( y_2 = xe^{r_1 x} \).  
   c. If \( r_1 \) and \( r_2 \) are complex, conjugate solutions: \( \alpha \pm \beta i \) then \( y_1 = e^{\alpha x} \cos \beta x \) and \( y_2 = e^{\alpha x} \sin \beta x \) |
METHODS FOR FINDING TWO LINEARLY INDEPENDENT SOLUTIONS (cont.)

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| Variable Coefficients, (Cauchy-Euler) | $ax^2y'' + bxy' + cy = 0$  $x > 0$ | 1. Substitute $y = x^m$ into the differential equation. It simplifies to $am^2 + (b - a)m + c = 0$. If $m$ is a solution to the characteristic equation then $y = x^m$ is a solution to the differential equation and
   a. If $m_1$ and $m_2$ are two real, distinct roots of characteristic equation then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$
   b. If $m_1 = m_2$ then $y_1 = x^m$ and $y_2 = x^m \ln x$.
   c. If $m_1$ and $m_2$ are complex, conjugate solutions $\alpha \pm \beta i$ then $y_1 = x^\alpha \cos(\beta \ln x)$ and $y_2 = x^\alpha \sin(\beta \ln x)$ |

Example #1. Solve the differential equation: $2t^2y'' + ty' - 3y = 0$, given that $y_1(t) = t^{-1}$ is a solution.

Solution:

Let $y(t) = v(t) \cdot y_1(t) = v \cdot t^{-1}$

$y'(t) = v'(t) \cdot y_1(t) + v(t) \cdot y_1'(t) = v' \cdot t^{-1} - v \cdot t^{-2}$

$y''(t) = v''(t) \cdot y_1(t) + 2v'(t) \cdot y_1'(t) + v(t) \cdot y_1''(t) = v'' \cdot t^{-1} - 2v' \cdot t^{-2} + 2vt^{-3}$

$2t^2y'' + ty' - 3y = 0 \implies 2t^2(v'' \cdot t^{-1} - 2v' \cdot t^{-2} + 2vt^{-3}) + (v' \cdot t^{-1} - v \cdot t^{-2}) - 3(v \cdot t^{-1}) = 0$

$2tv'' - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} = 0$

$2tv'' - 3v' = 0$

Let $v' = u$ so $v'' = u'$ then

$2tv'' - 3v' = 0 \implies 2tu' - 3u = 0$

$u' - \frac{3}{2t} u = 0$ (First order linear equation)

$u = t^{3/2} \implies v = \frac{2}{5} t^{5/2}$, at this point we can ignore the constant coefficients so take $v = t^{5/2}$

Substitute $v$ back into $y(t) = v(t) \cdot y_1(t)$ to get the second linearly independent solution.

$y_2 = v \cdot y_1 = t^{5/2} \cdot t^{-1} = t^{3/2}$

The general solution is:

$$y = C_1y_1 + C_2y_2$$

$$y = C_1t^{-1} + C_2t^{3/2}$$
Example #2. Solve the differential equation: \( y'' - 2y' + y = 0 \)

Solution:

Characteristic equation: 
\[
\begin{align*}
0 & = r^2 - 2r + 1 = 0 \\
(r-1)^2 & = 0 \\
r & = 1, 1 \quad \text{(Repeated roots)}
\end{align*}
\]

\( \Rightarrow y_1 = C_1e^x \) and \( y_2 = C_2xe^x \)

So the general solution is: 
\[
y = C_1e^x + C_2xe^x
\]

Example #3. Solve the differential equation: \( t^2 y''(t) - 4ty'(t) + 4y(t) = 0 \), given that \( y(1) = -2 \), \( y'(1) = -11 \)

Solution: The substitution: \( y = t^m \) yields to the characteristic equation:
\[
\begin{align*}
m^2 + (-4 - 1)m + 4 & = 0 \\
m^2 - 5m + 4 & = 0 \\
(m - 4)(m - 1) & = 0 \\
m & = 4 \text{ or } m = 1 \quad \text{two distinct, real solutions}
\end{align*}
\]

So the solutions are: \( t^4 \) and \( t \). The general solution is
\[
y = C_1t^4 + C_2t
\]

Use \( y(1) = -2 \), \( y'(1) = -11 \) to find the solution to the initial value problem:
\[
\begin{align*}
y(1) & = -2 \quad \Rightarrow C_1 + C_2 = -2 \\
y'(1) & = -11 \quad \Rightarrow 4C_1 + C_2 = -11
\end{align*}
\]

Solving the system of linear equations gives us \( C_1 = -3 \) and \( C_2 = 1 \)

So the solution to the Initial Value Problem is
\[
y = t - 3t^4
\]

You try it:

1. Given that \( y_1(x) = e^{2x} \) is a solution of the following differential equation \( 9y'' - 12y' + 4y = 0 \). Use the reduction of order to find a second solution.
   (Hint: \( y'' = 0 \) implies \( y' = 1 \))

   Find the general solution of the given second-order differential equations:
2. \( 3y'' + 2y' + y = 0 \)
3. \( x^2 y'' + 5xy' + 4y = 0 \)
Solutions:

#1: \( y_2 = x e^{2/3} \)

#2: \( y = e^{-2/3} \left[ C_1 \cos \left( \frac{\sqrt{2}}{3} x \right) + C_2 \sin \left( \frac{\sqrt{2}}{3} x \right) \right] \)

#3: \( y = C_1 x^{-2} + C_2 x^{-2} \ln x \)