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Ramsey Numbers of C_4 versus Wheels and Stars

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Abstract Let $ex(n, C_4)$ denote the maximum size of a C_4-free graph of order n. For an even integer or odd prime power q, we prove that $ex(q^2 + q + 2, C_4) < \frac{1}{2}(q + 1)(q^2 + q + 2)$, which leads to an improvement of the upper bound on Ramsey numbers $R(C_4, W_{q^2+2})$, where W_n is a wheel of order n. By using a simple polarity graph G_q for a prime power q, we construct the graphs whose complements do not contain $K_{1,m}$ or W_m, and then determine some exact values of $R(C_4, K_{1,m})$ and $R(C_4, W_m)$. In particular, we prove that $R(C_4, K_{1,q^2-2}) = q^2 + q - 1$ for $q \geq 3$, $R(C_4, W_{q^2-1}) = q^2 + q - 1$ for $q \geq 5$, and $R(C_4, W_{q^2+2}) = q^2 + q + 2$ for $q \geq 7$.

Keywords Ramsey number · Wheel · Cycle · Extremal graph · Polarity graph

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex-set $V(G)$ and edge-set $E(G)$, $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G, and $G \backslash S$ is the subgraph induced by the set $V(G) - S$. For $v \in S$, define $N_{G[S]}(v) = \{u : u \in S \wedge uv \in E(G)\}$ and $d_{G[S]}(v) = |N_{G[S]}(v)|$. If

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$S = V(G)$, we simply write $N(v)$, $d(v)$, and $N[v] = N(v) \cup \{v\}$. $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degree of G, respectively. C_k is a cycle of length k, $K_{1,n}$ is a star graph of order $n+1$, and W_{m+1} is a wheel with m spokes.

We use $ex(n, C_4)$ to denote the maximum size of a C_4-free graph of order n. A C_4-free graph of size $ex(n, C_4)$ is called an *extremal graph*, and let $EX(n, C_4)$ denote the set of all corresponding extremal graphs. Clapham, Flockhart and Sheehan [3] obtained the exact values of $ex(n, C_4)$ and the graphs in $EX(n, C_4)$ for $n \leq 21$. Yang and Rowlinson [15] determined the exact values of $ex(n, C_4)$ for $22 \leq n \leq 31$ and the corresponding extremal graphs. Recently, Shao, Xu and Xu [13] established that $ex(32, C_4) = 92$. In [12], Reiman determined the upper bound $ex(n, C_4) < \frac{1}{4}n(1 + \sqrt{4n-3})$ for $n \geq 4$. Füredi [6,7] determined that $ex(q^2+q+1, C_4) \leq \frac{1}{2}q(q+1)^2$ for all $q > 13$, and that the equality holds for all prime powers q. Firke et al. [5] showed that $ex(q^2+q, C_4) \leq \frac{1}{2}q(q+1)^2 - q$ for even q, and that the equality holds for $q = 2^k$. The main result of this paper related to $ex(n, C_4)$ is as follows.

Theorem 1 *If q is even or an odd prime power, then*

$$ex\left(q^2+q+2, C_4\right) < \frac{1}{2}(q+1)\left(q^2+q+2\right).$$

The Ramsey arrowing operator $\rightarrow$ is a logical predicate, which holds for graphs $G \rightarrow (H_1, H_2)$ if and only if for all partitions of the edges of G into two colors G_1 and G_2 there exists $H_1 \subseteq G_1$ or $H_2 \subseteq G_2$. The Ramsey number $R(H_1, H_2)$ is the smallest n such that $K_n \rightarrow (H_1, H_2)$. An $(H_1, H_2; n)$-*graph* denotes any graph not containing H_1 and not containing H_2 in the complement. In 1989, Burr et al. [2] showed that $m+\sqrt{m}-6m^{11/40} \leq R(C_4, K_{1,m}) \leq m+\lceil\sqrt{m}\,\rceil+1$. Parsons [10] proved that $R(C_4, K_{1,q^2}) = q^2+q+1$ and $R(C_4, K_{1,q^2+1}) = q^2+q+2$. For the Ramsey numbers of C_4 versus wheels, it is known that $R(C_4, W_4) = 10$, $R(C_4, W_5) = 9$ and $R(C_4, W_6) = 10$ [11]. Tse [14] determined the exact values of $R(C_4, W_m)$ for $7 \leq m \leq 13$. Dybizbański and Dzido [4] proved that $R(C_4, W_m) = m+4$ for $14 \leq m \leq 17$ and $R(C_4, W_{q^2+1}) = q^2+q+1$ for prime power $q \geq 4$. They also gave an upper bounds on $R(C_4, W_m)$ for $m \geq 11$ [see Theorem 7(d)]. We extend these results by our Theorems 2 and 4.

Theorem 2 *If q is even or an odd prime power, and $q \geq 7$, then*

$$R\left(C_4, W_{q^2+2}\right) \leq q^2+q+2.$$

For a prime power q, Abreu, Balbuena and Labbate [1] described *a simple polarity graph* G_q. We construct new graphs based on G_q whose complements do not contain $K_{1,m}$ or W_m, and then determine some exact values of $R(C_4, K_{1,m})$ or $R(C_4, W_m)$, as follows.

Theorem 3 *If $q \geq 3$ is a prime power, then*

(a) $R(C_4, K_{1,q^2-2}) = q^2+q-1$, *and*
(b) $R(C_4, K_{1,q^2-k-1}) = q^2+q-k$ *for even q, where $0 \leq k \leq q$ except $k \in \{1, q-1\}$.*

Theorem 4 *If $q \geq 3$ is a prime power, then*

(a) $R(C_4, W_{q^2+2}) = q^2 + q + 2$ *for* $q \geq 7$,
(b) $R(C_4, W_{q^2-1}) = q^2 + q - 1$, *and*
(c) $R(C_4, W_{q^2-k}) = q^2 + q - k$ *for even* q, *where* $0 \leq k \leq q$ *except* $k \in \{1, q-1\}$.

As mentioned previously, it was shown that Theorem 4(b) and (c) hold for $q = 3$ and 4 in [4,14]. Some well known results which will be used in our proofs are summarized in the next three theorems.

Theorem 5 [9] *Let G be a graph of order $n \geq 3$. If $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u and v, then G is Hamiltonian.*

Theorem 6 [12] $ex(n, C_4) < \frac{1}{4}n(1 + \sqrt{4n-3})$ *for* $n \geq 4$.

Theorem 7 [4,8,10]

(a) $R(C_4, C_n) = n + 1$ *for* $n \geq 6$,
(b) $R(C_4, K_{1,q^2+1}) = q^2 + q + 2$ *for any prime power* q,
(c) $R(C_4, K_{1,m}) \leq m + \lceil \sqrt{m} \rceil + 1$ *for* $m \geq 2$, *and*
(d) $R(C_4, W_m) \leq m + \lfloor \sqrt{m-2} \rfloor + 1$ *for* $m \geq 11$.

2 Proof of Theorem 1

Lemma 8 *Let q be an even integer or odd prime power. If G is a graph of order $q^2 + q + 2$ such that $\delta(G) \geq q + 1$, then $C_4 \subseteq G$.*

Proof If q is an odd prime power, then since $\delta(G) \geq q + 1$, we have $\Delta(\overline{G}) \leq q^2$, and thus $K_{1,q^2+1} \nsubseteq \overline{G}$. By Theorem 7(b), it follows that $C_4 \subseteq G$. If q is even, there are two cases depending on $\delta(G)$.

Case 1. Suppose that $\delta(G) \geq q + 2$, then

$$|E(G)| \geq \frac{1}{2}(q+2)\left(q^2+q+2\right).$$

Since

$$q + 2 = \frac{1}{2}\left(1 + \sqrt{4(q+3/2)^2}\right) > \frac{1}{2}\left(1 + \sqrt{4(q^2+q+2)-3}\right),$$

we have

$$|E(G)| > \frac{1}{4}\left(q^2+q+2\right)\left(1 + \sqrt{4(q^2+q+2)-3}\right).$$

By Theorem 6, we have $C_4 \subseteq G$, it completes Case 1.

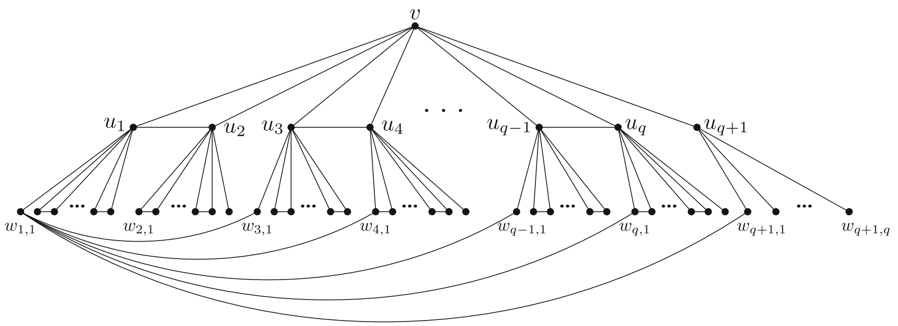

Fig. 1 Graph G in Case 2 of Lemma 8

Case 2. Suppose that $\delta(G) = q + 1$. Assume that $C_4 \nsubseteq G$ and consider $v \in V(G)$ such that $d(v) = q+1$, and $N(v) = \{u_i : 1 \le i \le q+1\}$. Each vertex u_i is adjacent to at most one vertex of $N(v)$, since otherwise $C_4 \subseteq G$. Note that $q + 1$ is odd, and thus, there exists at least one vertex of $N(v)$ which is nonadjacent to any vertex of $N(v)$, say u_{q+1}. Therefore, since $\delta(G) = q + 1$ and $C_4 \nsubseteq G$, we have u_{q+1} is adjacent to at least q vertices in $V(G) - N[v]$, and each vertex u_i, for $1 \le i \le q$, is adjacent to at least $q - 1$ vertices in $V(G) - N[v]$. Thus $|\bigcup_{i=1}^{q+1} N(u_i) - N[v]| \ge q(q - 1) + q = q^2$. Since $C_4 \nsubseteq G$, there does not exist any vertex in $V(G)-N[v]$ adjacent to two vertices of $N(v)$. Considering that $|V(G) - N[v]| = q^2$, we have that u_{q+1} is adjacent to q vertices in $V(G)-N[v]$, say $w_{q+1,j}$ for $1 \le j \le q$, and each vertex u_i, for $1 \le i \le q$, is adjacent to $q - 1$ vertices in $V(G) - N[v]$, say $w_{i,j}$ for $1 \le j \le q - 1$. Therefore, each vertex u_i for $1 \le i \le q$ has to be adjacent to exactly one vertex of $N(v)-\{u_{q+1}\}$, say $u_j u_{j+1} \in E(G)$ for odd j, $1 \le j \le q - 1$. Now we consider the vertices $w_{1,j}$ for $1 \le j \le q-1$. Similarly, since $q-1$ is odd and $C_4 \nsubseteq G$, there exists at least one vertex $w_{1,j}$ for $1 \le j \le q - 1$, which is nonadjacent to some other vertex of $w_{1,j}$, say $w_{1,1}$. Since $C_4 \nsubseteq G$, $w_{1,1}$ is nonadjacent to any vertex $w_{2,j}$ for $1 \le j \le q - 1$, and there is at most one vertex $w_{i,j}$ adjacent to $w_{1,1}$ for each i, $3 \le i \le q + 1$. Without loss of generality, let $w_{1,1}w_{i,1} \in E(G)$ for $3 \le i \le q + 1$ as shown in Fig. 1. Hence we have $d(w_{1,1}) = q$, a contradiction with $\delta(G) = q + 1$, thus Case 2 and the lemma hold. □

Proof of Theorem 1. For an even integer or odd prime power q, by Lemma 8, there doesn't exist a C_4-free graph of order $q^2 + q + 2$ such that $\delta(G) \ge q + 1$. Hence we have $ex(q^2 + q + 2, C_4) < \frac{1}{2}(q + 1)(q^2 + q + 2)$. □

3 The Upper Bounds on $R(C_4, W_{q^2+2})$

Proof of Theorem 2 For an even integer or odd prime power $q \ge 7$, suppose that $R(C_4, W_{q^2+2}) > q^2 + q + 2$, and let G be a $(C_4, W_{q^2+2}; q^2 + q + 2)$-graph. By Lemma 8 we have $\delta(G) \le q$. There are two cases depending on $\delta(G)$.

Case 1. Suppose that $\delta(G) \le q - 1$. Let $v \in V(G)$ be such that $d(v) \le q - 1$, then $d_{\overline{G}}(v) \ge q^2 + 2$. Since $C_4 \nsubseteq G$ and $R(C_4, C_{q^2+1}) = q^2 + 2$ by Theorem 7(a), we have $W_{q^2+2} \subseteq \overline{G}$, a contradiction.

Case 2. Suppose that $\delta(G) = q$. Let $v \in V(G)$ be a vertex with $d(v) = q$, $H = G[N_{\overline{G}}(v)]$, and let v_1 and v_2 be any two vertices such that $v_1v_2 \in E(H)$. Then we have $|V(H)| = q^2 + 1$, and

$$d_{\overline{H}}(v_1) + d_{\overline{H}}(v_2) = 2q^2 - (d_H(v_1) + d_H(v_2)) . \tag{1}$$

Note that G has at most $ex(q^2 + q + 2, C_4)$ edges. Let e be defined by

$$ex\left(q^2 + q + 2, C_4\right) = \left\lceil \frac{(q^2 + q + 2)\delta(G)}{2} \right\rceil + e. \tag{2}$$

For $v_1v_2 \in E(H)$, by the same argument as in [4], we have

Claim 1. $d_H(v_1) + d_H(v_2) \leq 2\delta(G) + e + 1$.

On the other hand, the upper bound on e is obtained in the next claim.

Claim 2. $e \leq q^2 - 2q - 2$ for $q \geq 7$.

Proof of Claim 2. By Theorem 6, we have

$$\begin{aligned} e &= ex\left(q^2 + q + 2, C_4\right) - \left\lceil \frac{(q^2 + q + 2)\delta(G)}{2} \right\rceil \\ &< \frac{1}{4}\left(q^2 + q + 2\right)\left(1 + \sqrt{4(q^2 + q + 2) - 3}\right) - \frac{1}{2}\left(q^2 + q + 2\right) q \\ &< \frac{1}{4}\left(q^2 + q + 2\right)\left(1 + 2q + \frac{5}{4}\right) - \frac{1}{2}\left(q^2 + q + 2\right) q \\ &= \frac{9}{16}\left(q^2 + q + 2\right). \end{aligned}$$

Since $q \geq 7$, we have $e \leq q^2 - 2q - 2$, thus Claim 2 holds.

By equality (1) and Claim 1, we have $d_{\overline{H}}(v_1) + d_{\overline{H}}(v_2) \geq 2q^2 - (2\delta(G) + e + 1) = 2q^2 - 2q - e - 1$. By Claim 2, we have $d_{\overline{H}}(v_1) + d_{\overline{H}}(v_2) \geq q^2 + 1$. By Theorem 5 we have $C_{q^2+1} \subseteq \overline{H}$, thus v together with the vertices of $V(\overline{H})$ would form a W_{q^2+2} in $\overline{G}$, a contradiction. Hence the assumption does not hold in Case 2.

This completes the proof of Theorem 2. □

Theorem 2 implies the following.

Corollary 9 *If q is a prime power and $q \geq 7$, then $R(C_4, W_{q^2+2}) \leq q^2 + q + 2$.*

4 The Ramsey Numbers of $R(C_4, K_{1,m})$ and $R(C_4, W_m)$

There are many old descriptions of polarity graphs. We choose to use the notation of [1] which is particularly suitable for our purpose. In this paper, Abreu, Balbuena and Labbate presented the adjacency matrix M° of the polarity graph $\widehat{G}_q$ from a projective plane $PG(2, q)$ for a prime power q. The graph $\widehat{G}_q$ has $q^2 + q + 1$ vertices and $q + 1$ loops. The matrix M° is J_2-free, where J_2 is a matrix of order 2 all of whose entries are 1, so $\widehat{G}_q$ is a C_4-free graph. The simple polarity graph G_q with the matrix M^* is the graph obtained from $\widehat{G}_q$ by deleting all $q+1$ loops. The graph G_q has q^2+q+1 vertices, in which $q+1$ vertices have degree q and all other vertices have degree $q+1$. It has triangles, but no C_4 and it has diameter two. Taking $q=2, 3$, and 4 as examples, the matrices of G_q are shown in Table 1. We label the vertices of G_q from v_1 to v_{q^2+q+1} according to the rows (or columns) of M^*, that is, $V(G_q) = \{v_1, v_2, \ldots, v_{q^2+q+1}\}$.

We notice that the symmetric matrices M^* of G_q have the following properties.

Fact 10 (1) $M^*\left[q^2+i, q^2+q+1\right] = 1$ *for* $1 \le i \le q$. *Thus*

$$N\left(v_{q^2+q+1}\right) = \left\{v_{q^2+1}, v_{q^2+2}, \ldots, v_{q^2+q}\right\}.$$

Table 1 The adjacency matrices M^* of graphs G_2, G_3, and G_4

(**a**) G_2

v_1	0	1	1	0	1	0	0
v_2	1	0	0	1	1	0	0
v_3	1	0	0	0	0	1	0
v_4	0	1	0	0	0	1	0
v_5	1	1	0	0	0	0	1
v_6	0	0	1	1	0	0	1
v_7	0	0	0	0	1	1	0

(**b**) G_3

v_1	0	1	0	0	0	1	1	0	0	1	0	0	0
v_2	1	0	0	0	1	0	0	0	1	1	0	0	0
v_3	0	0	0	1	0	0	0	1	0	1	0	0	0
v_4	0	0	1	0	1	0	1	0	0	0	1	0	0
v_5	0	1	0	1	0	0	0	0	1	0	1	0	0
v_6	1	0	0	0	0	0	0	1	0	0	1	0	0
v_7	1	0	0	1	0	0	0	0	0	0	0	1	0
v_8	0	0	1	0	0	1	0	0	1	0	0	1	0
v_9	0	1	0	0	1	0	0	1	0	0	0	1	0
v_{10}	1	1	1	0	0	0	0	0	0	0	0	0	1
v_{11}	0	0	0	1	1	1	0	0	0	0	0	0	1
v_{12}	0	0	0	0	0	0	1	1	1	0	0	0	1
v_{13}	0	0	0	0	0	0	0	0	0	1	1	1	0

(**c**) G_4

v_1	0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0	1	0	0	0	0
v_2	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0
v_3	0	0	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0
v_4	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0
v_5	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0	0	0
v_6	0	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0	0	1	0	0	0
v_7	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0

Table 1 continued

v_8	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0
v_9	0	0	0	1	0	1	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0
v_{10}	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0
v_{11}	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	1	0	0
v_{12}	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	1	0	0
v_{13}	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0
v_{14}	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
v_{15}	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0
v_{16}	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0
v_{17}	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
v_{18}	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1
v_{19}	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1
v_{20}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1
v_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0

(2) *For each* $1 \leq i \leq q$, $M^*[(i-1)q + j, q^2 + i] = 1$ *for* $1 \leq j \leq q$. *Thus* $d(v_{q^2+i}) = q + 1$ *and*

$$N\left(v_{q^2+i}\right) = \left\{v_{(i-1)q+1}, v_{(i-1)q+2}, \ldots, v_{iq}\right\} \cup \left\{v_{q^2+q+1}\right\}.$$

(3) $M^*[(i-1)q+1, q^2-q+1] = 1$ *for* $1 \leq i \leq q-1$, *and* $M^*[q^2+q, q^2-q+1] = 1$. *Thus* $d(v_{q^2-q+1}) = q$ *and*

$$N\left(v_{q^2-q+1}\right) = \left\{v_1, v_{q+1}, v_{2q+1}, \ldots, v_{(q-2)q+1}\right\} \cup \left\{v_{q^2+q}\right\}.$$

(4) *For odd* q, *the main diagonal of the matrix* M° *has an entry* 1 *in the last position, and all the other* $1'$*s on the main diagonal are distributed one in each block of the main diagonal of blocks. Hence we see that each vertex of* $N(v_{q^2+q}) - \{v_{q^2-q+1}, v_{q^2+q+1}\}$ *has degree* $q + 1$, *that is,*

$$d(v_i) = q + 1, \quad v_i \in \left\{v_{(q-1)q+2}, v_{(q-1)q+3}, \ldots, v_{q^2}\right\}.$$

(5) *For even* q, *all 1's on the main diagonal of* M° *are in row* $q(q-1) + i$, *for* $1 \leq i \leq q$, *and in the last row. Hence we see that each vertex of* $N(v_{q^2+q})$ *has degree* q, *that is,*

$$d(w) = \begin{cases} q, & for\ w \in N(v_{q^2+q}), \\ q + 1, & for\ w \notin N(v_{q^2+q}). \end{cases}$$

To illustrate Fact 10 (4), we take $\widehat{G}_3$ as an example, the matrix M° is shown in Table 2, where all 1's on the main diagonal are underlined. By Fact 10 (2), the vertices

Table 2 The adjacency matrix M° of $\widehat{G}_3$

v_1	0	1	0	0	0	1	1	0	0	1	0	0	0
v_2	1	0	0	0	1	0	0	0	1	1	0	0	0
v_3	0	0	$\underline{1}$	1	0	0	0	1	0	1	0	0	0
v_4	0	0	1	0	1	0	1	0	0	0	1	0	0
v_5	0	1	0	1	0	0	0	0	1	0	1	0	0
v_6	1	0	0	0	0	$\underline{1}$	0	1	0	0	1	0	0
v_7	1	0	0	1	0	0	$\underline{1}$	0	0	0	0	1	0
v_8	0	0	1	0	0	1	0	0	1	0	0	1	0
v_9	0	1	0	0	1	0	0	1	0	0	0	1	0
v_{10}	1	1	1	0	0	0	0	0	0	0	0	0	1
v_{11}	0	0	0	1	1	1	0	0	0	0	0	0	1
v_{12}	0	0	0	0	0	0	1	1	1	0	0	0	1
v_{13}	0	0	0	0	0	0	0	0	0	1	1	1	$\underline{1}$

of $N(v_{q^2+q}) - \{v_{q^2+q+1}\}$ belong to one block. Note that v_{q^2-q+1} has a loop, and by Fact 10 (1), $d(v_{q^2+q+1}) = q$.

To illustrate Fact 10 (5), we take $\widehat{G}_4$ as an example, the matrix M° is shown in Table 3, where all 1's on the main diagonal are underlined. Note that each vertex of $N(v_{q^2+q})$ has a loop.

Lemma 11 *For $v \in V(G_q)$ with $d(v) = q$, each vertex of $N(v)$ has degree $q + 1$.*

Proof Assume there exists a vertex u such that $d(u) = q$ and $uv \in E(G_q)$. Since the vertex of degree q in G_q has a loop in $\widehat{G}_q$ [1], the vertices u, v have loops in $\widehat{G}_q$. Hence the matrix M° of $\widehat{G}_q$ would contain a submatrix J_2 of vertices u and v, a contradiction with $C_4 \nsubseteq \widehat{G}_q$. □

Lemma 12 *Let G be a C_4-free graph of order n.*

(a) *If $m > n - \delta(G) - 1$, then G is a $(C_4, K_{1,m})$-graph, and*
(b) *If $m > n - \delta(G)$, then G is a (C_4, W_m)-graph.*

Proof If $m > n - \delta(G) - 1$, then $\Delta(\overline{G}) < m$. So we have $K_{1,m} \nsubseteq \overline{G}$, thus (a) holds. If $m > n - \delta(G)$, then $\Delta(\overline{G}) < m - 1$. Hence we have $W_m \nsubseteq \overline{G}$, and thus (b) holds. □

In Construction 13–15 below we define graphs H_s of order s such that $\delta(H_s) = q$ and $C_4 \nsubseteq H_s$. The graphs H_s are constructed by removing some vertices from G_q. These graphs are used in the proofs of Theorems 3 and 4.

Construction 13 (H_{q^2+q-i} for even q, and $1 \le i \le q - 1$)

By Fact 10 (1) and 10 (2), we have $N(v_{q^2+q+1}) = \{v_{q^2+1}, v_{q^2+2}, \ldots, v_{q^2+q}\}$ and $d(v_i) = q + 1$ for $v_i \in N(v_{q^2+q+1})$. We set $H_{q^2+q} = G_q \backslash \{v_{q^2+q+1}\}$, thus the vertices $v_{q^2+1}, v_{q^2+2}, \ldots, v_{q^2+q}$ have degree q in H_{q^2+q}. By Fact 10 (2),

Table 3 The adjacency matrix M° of $\widehat{G}_4$

v_1	0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0	1	0	0	0	0
v_2	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0
v_3	0	0	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0
v_4	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0
v_5	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0	0	0
v_6	0	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0	0	1	0	0	0
v_7	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0
v_8	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0
v_9	0	0	0	1	0	1	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0
v_{10}	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0
v_{11}	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	1	0	0
v_{12}	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	1	0	0
v_{13}	1	0	0	0	1	0	0	0	1	0	0	0	$\underline{1}$	0	0	0	0	0	0	1	0
v_{14}	0	1	0	0	0	1	0	0	0	1	0	0	0	$\underline{1}$	0	0	0	0	0	1	0
v_{15}	0	0	1	0	0	0	1	0	0	0	1	0	0	0	$\underline{1}$	0	0	0	0	1	0
v_{16}	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	$\underline{1}$	0	0	0	1	0
v_{17}	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
v_{18}	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1
v_{19}	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1
v_{20}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1
v_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	$\underline{1}$

$E(G[N(v_{q^2+q+1})]) = \emptyset$. By Fact 10 (5), each vertex in $\{v_{q^2+1}, v_{q^2+2}, \dots, v_{q^2+q-1}\}$ is adjacent to only vertices of degree $q+1$ in H_{q^2+q}. We construct the graph H_{q^2+q-i} by removing one vertex v_{q^2+i}, namely

$$H_{q^2+q-i} = H_{q^2+q-i+1} \backslash \{v_{q^2+i}\}, \quad 1 \le i \le q-1.$$

Note that H_{q^2+1} is a q-regular graph.

Construction 14 (H_{q^2-1} for even q)

By Fact 10 (2) and 10 (5), $N(v_{q^2+q}) = \{v_{(q-1)q+1}, v_{(q-1)q+2}, \dots, v_{q^2}\} \cup \{v_{q^2+q+1}\}$, and $d(v_i) = q+1$ for $v_i \notin N(v_{q^2+q})$. Therefore, since $C_4 \nsubseteq G_q$, we have $N(v_i) \cap N(v_j) = \{v_{q^2+q}\}$ for $v_i, v_j \in N(v_{q^2+q})$. We set

$$H_{q^2-1} = G_q \backslash N[v_{q^2+q}].$$

Note that H_{q^2-1} is a q-regular graph.

Construction 15 (H_{q^2+q-2} for odd q)

By Fact 10 (1–3), there doesn't exist any vertex in G_q adjacent to two vertices of $\{v_{(q-1)q+1}, v_{q^2+q}, v_{q^2+q+1}\}$. By Fact 10 (4), each vertex of $N(v_{q^2+q}) -$

$\{v_{(q-1)q+1}, v_{q^2+q+1}\}$ has degree $q+1$. By Lemma 11 each vertex of $N(v_{(q-1)q+1})$ has degree $q+1$, so does each vertex of $N(v_{q^2+q+1})$. We set

$$H_{q^2+q-2} = G_q \backslash \{v_{(q-1)q+1}, v_{q^2+q}, v_{q^2+q+1}\}.$$

Proof of Theorem 3. By Theorem 7(c), we have the upper bounds in (a) and (b). So it is sufficient to prove the lower bounds. By Construction 13 of H_{q^2+q-2} for even $q \geq 4$, by Construction 15 of H_{q^2+q-2} for odd $q \geq 3$, and Lemma 12(a), we have $R(C_4, K_{1,q^2-2}) \geq q^2+q-1$, thus (a) holds.

For even q, using H_{q^2+q-i} as in Construction 13 for $1 \leq i \leq q-1$ and $i \neq 2$, H_{q^2-1} in Construction 14, and Lemma 12(a), we have $R(C_4, K_{1,q^2-k-1}) \geq q^2+q-k$ for even $q \geq 4$, where $0 \leq k \leq q$ except $k \in \{1, q-1\}$. Hence (b) holds. □

Proof of Theorem 4. Since G_q is a graph of order q^2+q+1 such that $C_4 \nsubseteq G_q$ and $\delta(G_q) = q$, by Lemma 12(b), we have $R(C_4, W_{q^2+2}) \geq q^2+q+2$. By Corollary 9, we have $R(C_4, W_{q^2+2}) \leq q^2+q+2$ for $q \geq 7$, and thus (a) holds.

By Theorem 7(d), we have an upper bounds on $R(C_4, W_m)$ in (b) and (c). So it is sufficient to prove the lower bounds. Using H_{q^2+q-2} as in Construction 13 for even $q \geq 4$, H_{q^2+q-2} as in Construction 15 for odd $q \geq 3$, and Lemma 12(b), we have $R(C_4, W_{q^2-1}) \geq q^2+q-1$, and thus (b) holds.

For even q, by Construction 13 of H_{q^2+q-i} for $1 \leq i \leq q-1$ and $i \neq 2$, by Construction 14 of H_{q^2-1}, and Lemma 12(b), we have $R(C_4, W_{q^2-k}) \geq q^2+q-k$ for even $q \geq 4$, where $0 \leq k \leq q$ except $k \in \{1, q-1\}$. Hence (c) holds. □

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