Some Computational and Theoretical Problems for Ramsey Numbers

bounds, constructions and connectivity

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Ramsey Numbers

- $R(G, H) = n$ iff
  minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color.

- $2 - colorings \cong graphs$, $R(m, n) = R(K_m, K_n)$
- Generalizes to k colors, $R(G_1, \cdots, G_k)$

- Theorem (Ramsey 1930): Ramsey numbers exist

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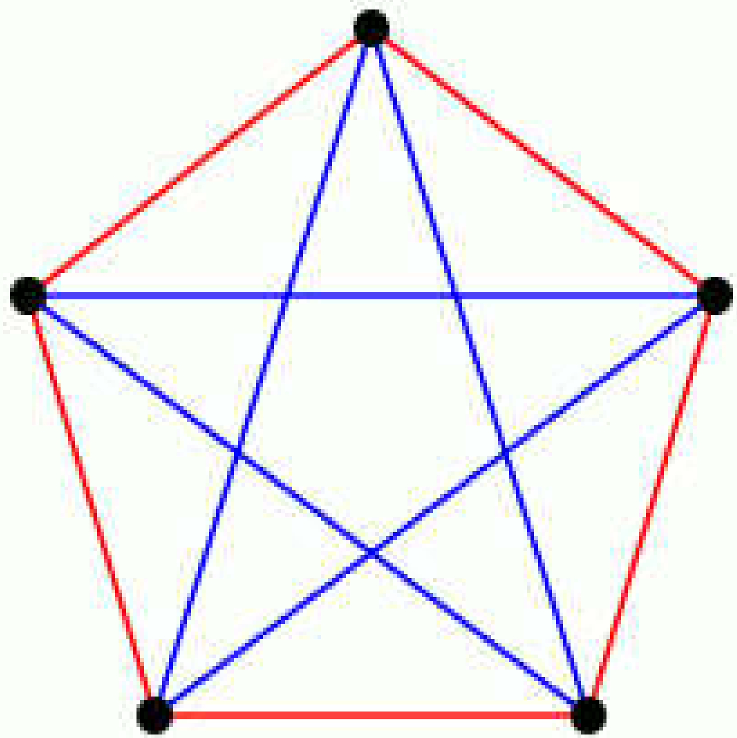
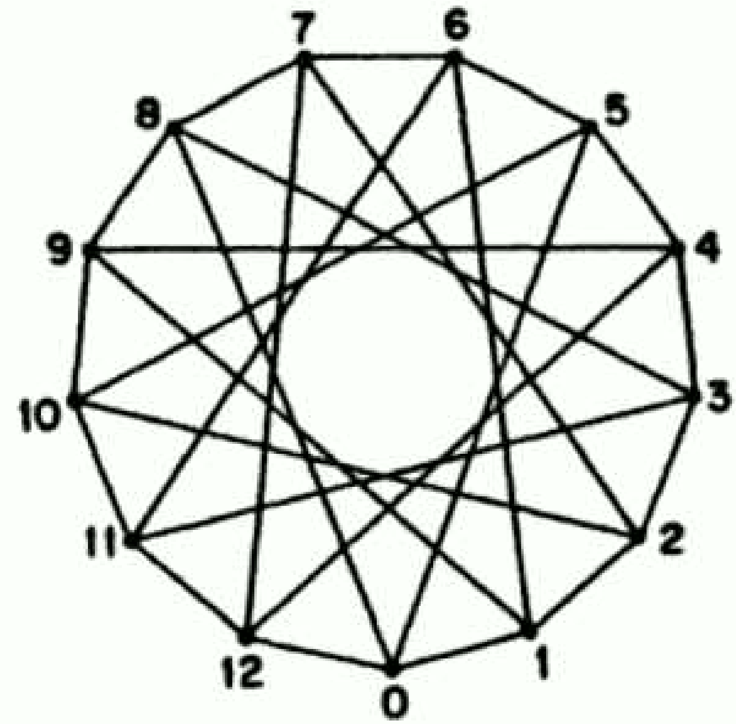
Unavoidable classics



$R(3,3) = 6$



$R(3,5) = 14$ [GRS'90]

Asymptotics
diagonal cases

- **Bounds** (Erdős 1947, Spencer 1975; Conlon 2010)

$$\frac{\sqrt{2}}{e} 2^{n/2} n < R(n,n) < R(n+1,n+1) \leq \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, $100)
  $\lim_{n\to\infty} R(n,n)^{1/n}$ exists.
  If it exists, it is between $\sqrt{2}$ and 4 ($250 for value).

- **Theorem** (Chung-Grinstead 1983)
  $L = \lim_{k\to\infty} R_k(3)^{1/k}$ exists.

  $3.199 < L$, (Fredricksen-Sweet 2000, X-Xie-Exoo-R 2004)

Asymptotics

Ramsey numbers avoiding K_3

- Kim 1995, lower bound
  Ajtai-Komlós-Szemerédi 1980, upper bound

$$R(3,n) = \Theta\left(\frac{n^2}{\log n}\right)$$

- Bohman 2009, triangle-free process
  Bohman-Keevash 2013
  Fiz Pontiveros-Griffiths-Morris 2013

  Shearer 1983 (upper bound)

$$\left(\frac{1}{4} + o(1)\right) n^2/\log n \leq R(3,n) \leq (1 + o(1))n^2/\log n$$

Off-Diagonal Cases

upper bounds

- Erdős-Szekeres 1935 (implicit)

$$R(m,n) \le \binom{m+n-2}{m-1}$$

- Ajtai-Komlós-Szemerédi 1980, Graham-Rödl 1981
  for fixed $n \ge 3$ and large m

$$R(m,n) \le c^n m^{n-1}/(\log m)^{n-2}$$

Off-Diagonal Cases

fixed small avoided K_m

- Bohman triangle-free process - 2009
  probabilistic lower bound

$$R(4, n) = \Omega(n^{5/2} / \log^2 n)$$

$$R(4, n) = O(n^3 / \log^2 n)$$

- Kostochka, Pudlák, Rödl - 2010
  constructive lower bounds

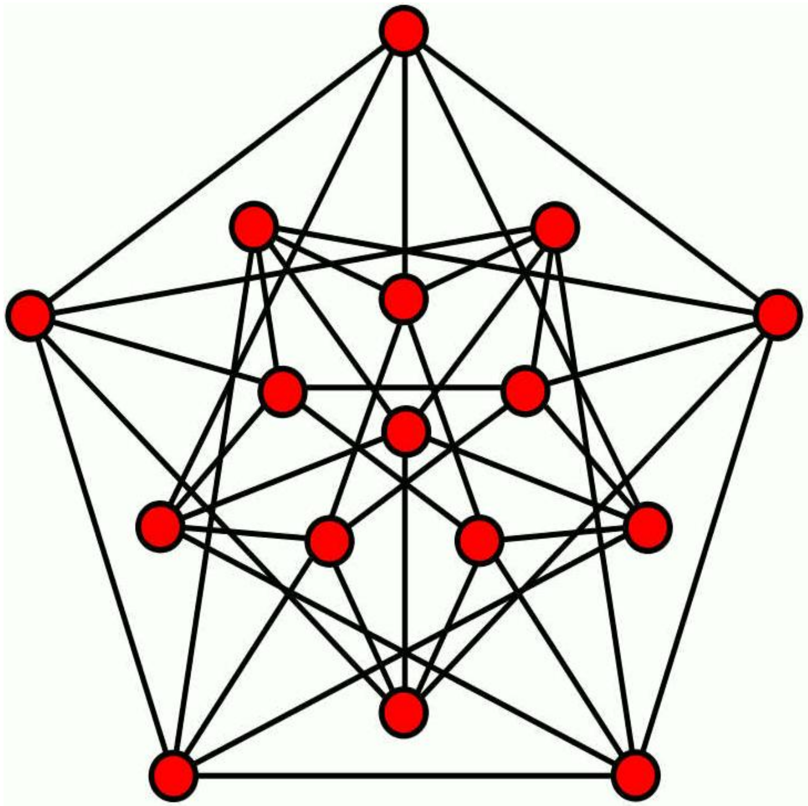
$$R(4, n) = \Omega(n^{8/5}), \quad R(5, n) = \Omega(n^{5/3}), \quad R(6, n) = \Omega(n^2)$$

(vs. probabilistic $5/2, 6/2, 7/2$ with /logs for 4, 5, 6)

Clebsch $(3,6;16)$-graph on $GF(2^4)$

$(x,y) \in E$ iff $x - y = \alpha^3$

[Wikipedia]



Alfred Clebsch (1833-1872)

#vertices / #graphs

no exhaustive searches beyond 13 vertices

3	4
4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
11	1018997864
12	165091172592
13	50502031367952 $\approx 5 * 10^{13}$
	too many to process
14	29054155657235488 $\approx 3 * 10^{16}$
15	31426485969804308768
16	64001015704527557894928
17	245935864153532932683719776
18	$\approx 2 * 10^{30}$

Compute or not

Arnold

Journal of Mathematical Fluid Mechanics, 2005

From the deductive mathematics point of view most of these results are not theorems, being only descriptions of several millions of particular observations. However, I hope that they are even more important than the formal deductions from the formal axioms, providing new points of view on difficult problems where no other approaches are that efficient.

Values and bounds on $R(k, l)$

two colors, avoiding K_k, K_l

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 42	47 50	52 59	59 68	66 77	73 87
4		18	25	36 41	49 61	59 84	73 115	92 149	100 191	128 238	136 291	146 349	155 417
5			43 49	58 87	80 143	101 216	133 316	148 442	171 633	194 848	218 1138	242 1461	267 1878
6				102 165	115 298	134 495	175 780	185 1171	253 1804	263 2566	317 3703	5033	401 6911
7					205 540	217 1031	242 1713	289 2826	405 4553	417 6954	511 10578	15263	22112
8						282 1870	317 3583	6090	10630	16944	817 27485	41525	861 63609
9							565 6588	581 12677	22325	38832	64864		
10								798 23556	45881	81123			1265

[ElJC survey *Small Ramsey Numbers*, revision #14, 2014, with numerous updates]

Bounds on $R(3, k) - R(3, k-1)$

Erdős and Sós, 1980, asked about

$3 \leq \Delta_k = R(3, k) - R(3, k-1) \leq k$:

$$\Delta_k \xrightarrow{k} \infty \,? \qquad \Delta_k / k \xrightarrow{k} 0 \,?$$

Look at $R(K_3, K_k - e)$ relative to $R(K_3, K_k) = R(3, k)$

$$\Delta_k = \big(R(K_3, K_k) - R(K_3, K_k - e)\big) + \big(R(K_3, K_k - e) - R(K_3, K_{k-1})\big)$$

$\Delta_k = R(3, k) - R(3, k-1)$

It is known that

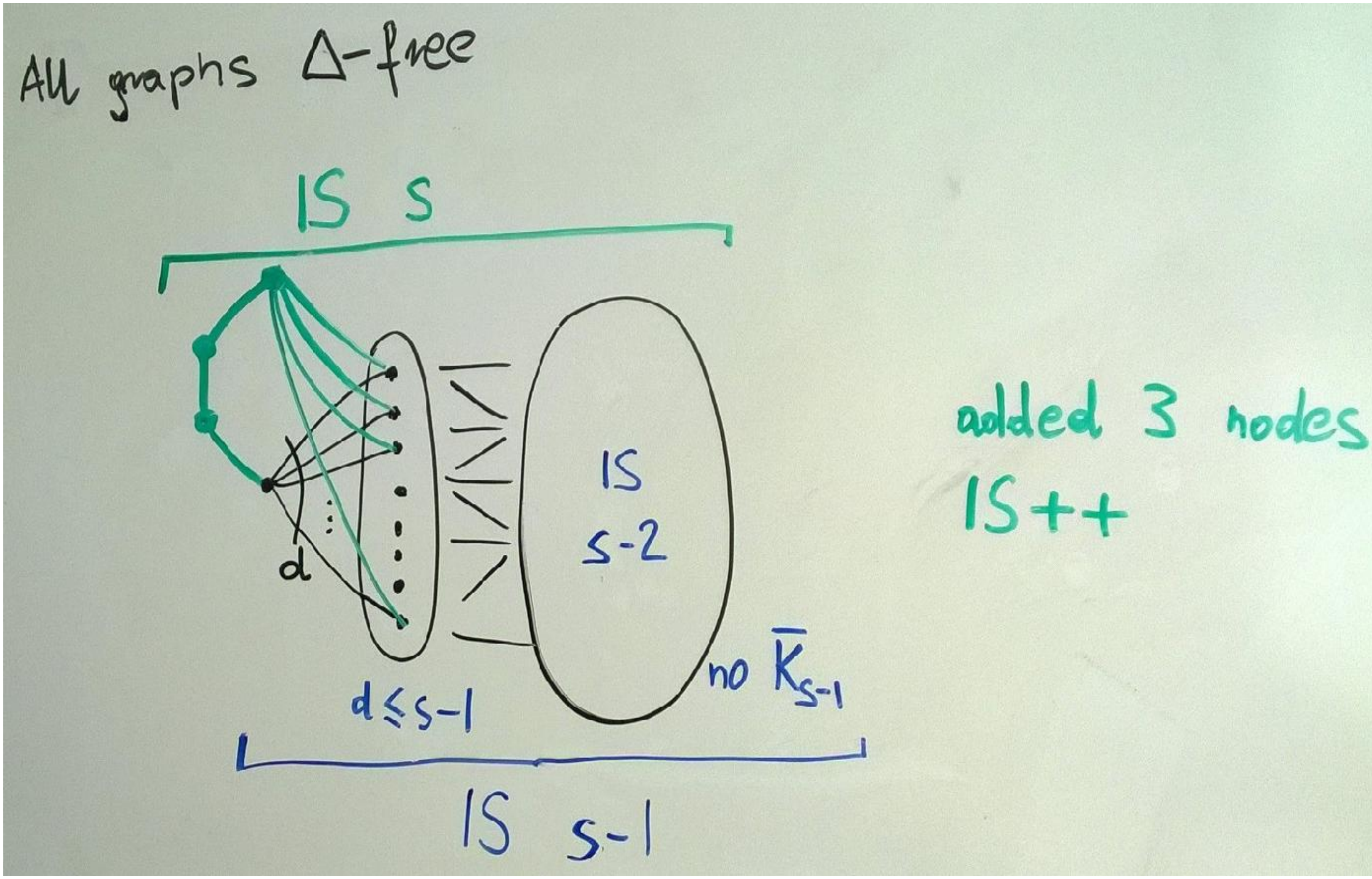
$$\left(\frac{1}{4} + o(1)\right)k^2/\log k \leq R(3, k) \leq (1 + o(1))k^2/\log k$$

All we have for K_k and $K_k - e$ is

(1) $3 \leq R(3, K_k) - R(3, K_{k-1}) \leq k$, easy old bounds,
(2) $R(3, K_{k-1}) \leq R(3, K_k - e) \leq R(3, K_k)$, trivial bounds,
(3) $4 \leq R(3, K_{k+1}) - R(3, K_k - e)$ (Zhu-Xu-R 2015).

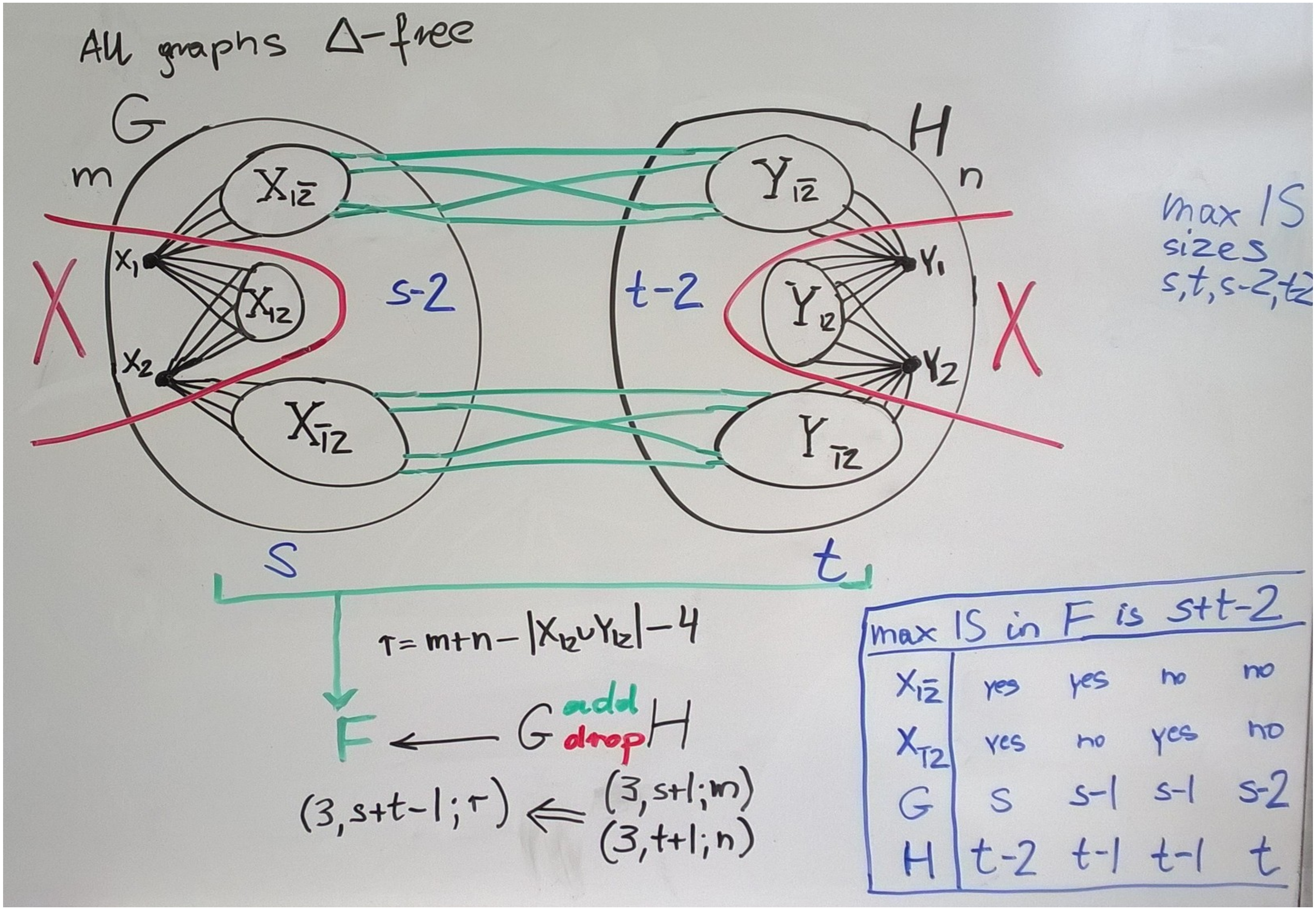
Problem. Improve over any of the inequalities in (1), (2) or (3), or their combination as (3) combines parts of (1) and (2).

$\Delta_s \geq 3$



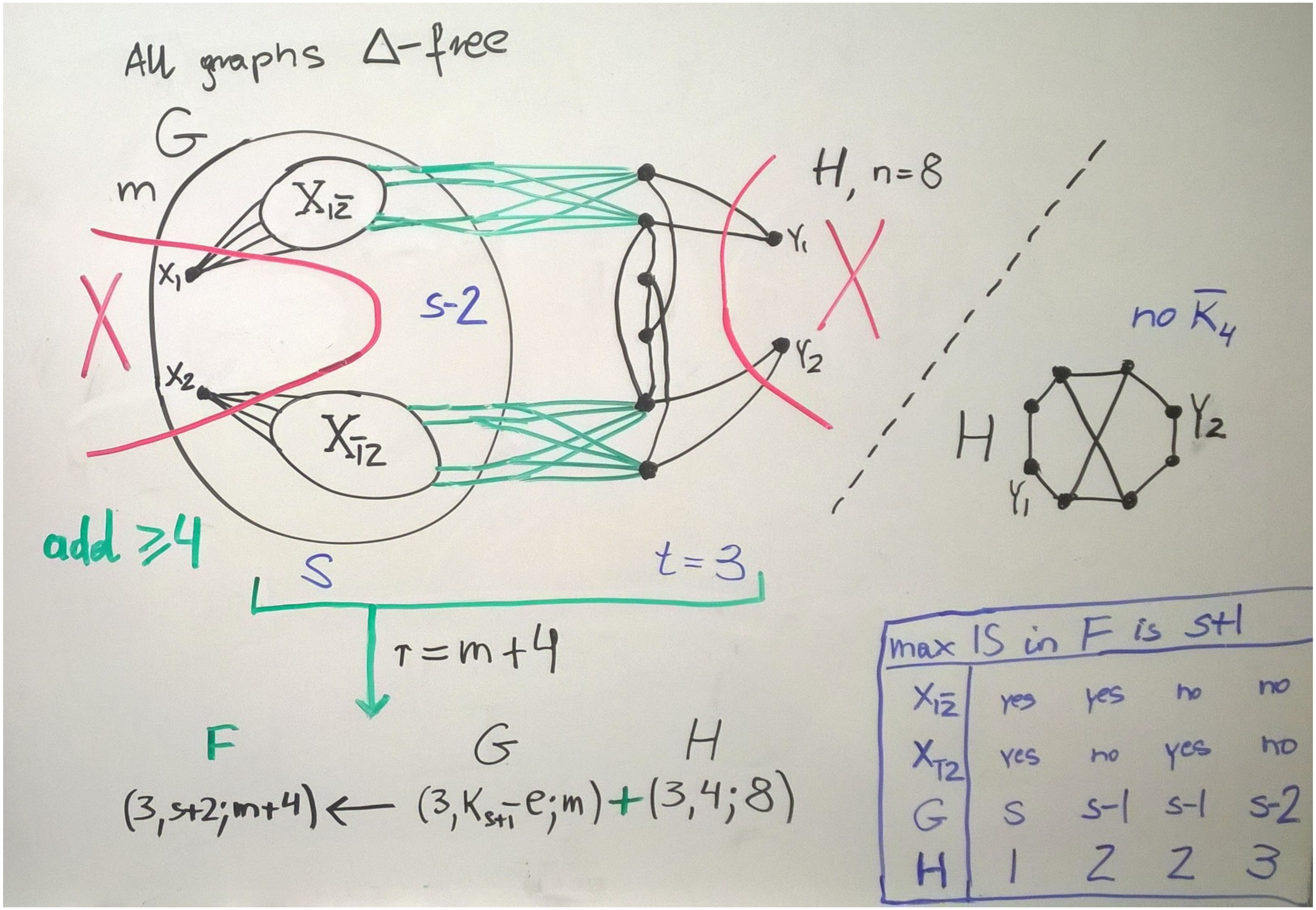
All graphs Δ-free
IS s
d
IS
s-2
added 3 nodes
IS++
no $\overline{K}_{s-1}$
d≤s-1
IS s-1

$R(3, s+t-1) \geq R(3, s+1) + R(3, t+1) - c_{st}$



$R(3, K_{k+1}) - R(3, K_k - e) \geq 4$

$k = s + 1$



Known bounds on $R(3, K_s)$ and $R(3, K_s - e)$

$J_s = K_s - e$

s	$R(3, J_s)$	$R(3, K_s)$	Δ_s	s	$R(3, J_s)$	$R(3, K_s)$	Δ_s
3	5	6	3	10	37	40–42	4–6
4	7	9	3	11	42–45	47–50	5–10
5	11	14	5	12	47–53	52–59	3–12
6	17	18	4	13	55–62	59–68	3–13
7	21	23	5	14	59–71	66–77	3–14
8	25	28	5	15	69–80	73–87	3–15
9	31	36	8	16	73–91	82–98	3–16

Table : $R(3, J_s)$ and $R(3, K_s)$, for $s \leq 16$ (Goedgebeur-R 2014).

Corollaries

from constructions

General, but perhaps not that strong

- $R(3, s+t) \geq R(3, s+1) + R(3, t+1) - 3$,
- $R(3, s+t-1) \geq R(3, K_{s+1} - e) + R(3, K_{t+1} - e) - 5$.

Applications, but quite interesting

- For $s \geq 3$ and $m = R(3, s+1) - 1$, if there exists a $(3, s+1; m)$-graph which is not bicritical, then $\Delta_{s+2} \geq 4$,
- $R(3, s+1) \geq R(3, K_s - e) + 4$.

Conjecture 1

and 1/2 of Erdős-Sós problem

Observe that
$R(3, s+k) - R(3, s-1) = \sum_{i=0}^{k} \Delta_{s+i}$.

We know that
$\Delta_s \geq 3$, $\Delta_s + \Delta_{s+1} \geq 7$, $\Delta_s + \Delta_{s+1} + \Delta_{s+2} \geq 11$.

Conjecture 1
There exists $d \geq 2$ such that $\Delta_s - \Delta_{s+1} \leq d$ for all $s \geq 2$.

Theorem
If Conjecture 1 is true, then $\lim_{s\to\infty} \Delta_s / s = 0$.

Conjecture 2

possibly easier to prove

Conjecture 2

There exists integer k such that

$$\lim_{s\to\infty} \sum_{i=0}^{k} \Delta_{s+i} = \infty.$$

Growth of Δ_s gives also insights on

- connectivity of Ramsey-critical graphs, Beveridge-Pikhurko 2008, Xu-Shao-R 2011
- hamiltonicity of Ramsey-critical graphs, Xu-Shao-R 2011
- chromatic gap, Gyárfás-Sebő-Trotignon 2012
- Shannon capacity of graphs, Xu-R 2013

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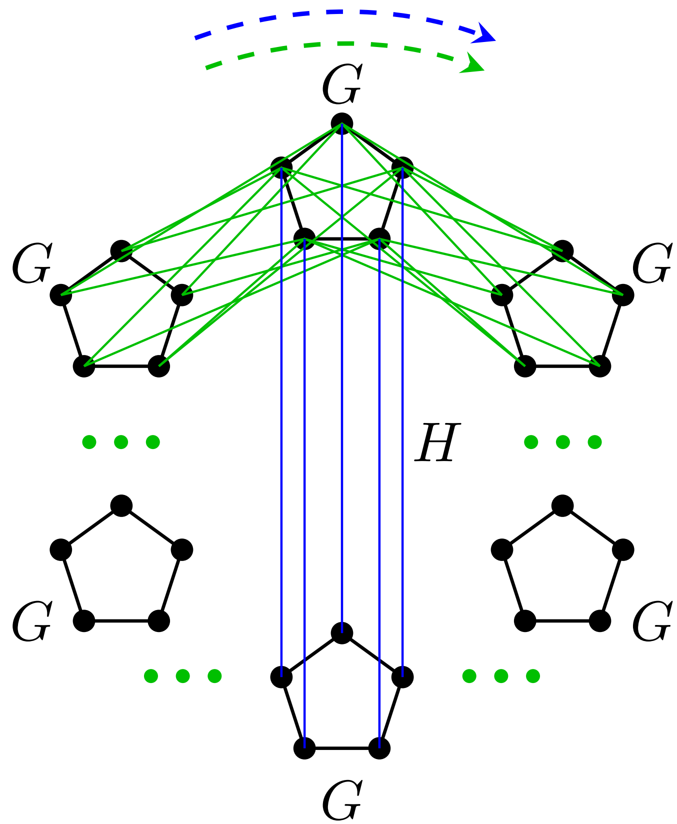
Construction

Chung-Cleve-Dagum 1993



Construction of $H \in \mathcal{R}(3,9;30)$ using $G = C_5 \in \mathcal{R}(3,3;5)$
$R(3,k) = \Omega(k^{\log 6/\log 4}) \approx \Omega(k^{1.29})$

Explicit $\Omega(k^{3/2})$ construction:
Alon 1994, Codenotti-Pudlák-Resta 2000

$R(5,5)$

year	reference	lower	upper	
1965	Abbott	38		quadratic residues in $\mathcal{Z}_{37}$
1965	Kalbfleisch		59	pointer to a future paper
1967	Giraud		58	LP
1968	Walker		57	LP
1971	Walker		55	LP
1973	Irving	42		sum-free sets
1989	Exoo	43		simulated annealing
1992	McKay-R		53	$(4,4)$-graph enumeration, LP
1994	McKay-R		52	more details, LP
1995	McKay-R		50	implication of $R(4,5) = 25$
1997	McKay-R		49	long computations
2014	McKay-Lieby			study of $(5,5;42)$-graphs

History of bounds on $R(5,5)$

$43 \leq R(5,5) \leq 49$

Conjecture. McKay-R 1997

$R(5,5) = 43$, and the number of $(5,5;42)$-graphs is precisely 656.

The known $(5,5;42)$-graphs have properties:

- # edges ranges from 423 to 438 (midpoint 430-431)
- mindeg 19, maxdeg 22
- no more than 2 symmetries
  232 graphs have only an involution
  424 graphs have trivial group
- none is almost regular

$R(5,5)$

McKay and Lieby, 2014

Define the distance between two graphs on n vertices to be k if their largest common induced subgraph has $n - k$ vertices.

- Very large computational effort to find distances between known $(5,5;42)$-graphs.
- Known $(5,5;42)$-graphs form one large cluster.
- McKay and Lieby report that any new $(5,5;42)$-graph H would have to be in distance at least 6 from every graph in the set of 656 known $(5,5;42)$-graphs.

What to do next?

Challenges

Theoretical

- better explicit constructive lower bounds for $R(3, k)$
- improve bounds for Δ_k, or a similar local difference
- generalize above beyond triangle-free graphs

Computational - improve any of the following

- $29 \leq R(C_5, K_8) \leq 33$
- $42 \leq R(3, K_{11} - e) \leq 45$
- Lower bounds for other larger parameters
- Other small puzzling $R(G, H)$, survey SRN 2014

What not to compute

infeasible without a breakthrough

Each of the following needs a new insight

- $R(3, 10) \leq 41$?
- $R(4, 6) \leq 40$?
- $R(5, 5) \leq 48$?

Seems hard to improve any of the corresponding lower bounds.

Growth of $R(m, n)$

Slow on citing this result...

In 1980, Paul Erdős wrote

Faudree, Schelp, Rousseau and I needed recently a lemma stating

$$\lim_{n\to\infty} \frac{r(n+1,n) - r(n,n)}{n} = \infty.$$

We could prove it without much difficulty, but could not prove that $r(n+1,n) - r(n,n)$ increases faster than any polynomial of n. We of course expect

$$\lim_{n\to\infty} \frac{r(n+1,n)}{r(n,n)} = C^{\frac{1}{2}},$$

where $C = \lim_{n\to\infty} r(n,n)^{1/n}$.

The best known bound for $r(n+1,n) - r(n,n)$ is $\Omega(n)$.

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Connectivity of Ramsey graphs

Xu-Shao-R 2011

Theorem.

If $k \geq 5$ and $s \geq 3$, then the connectivity of any Ramsey-critical (k, s)-graph is no less than k.
(improves by 1 Beveridge-Pikhurko 2008)

Theorem.

If $k \geq s - 1 \geq 1$ and $k \geq 3$, except $(k, s) = (3, 2)$, then any Ramsey-critical (k, s)-graph is Hamiltonian.

In particular, for $k \geq 3$, all diagonal Ramsey-critical (k, k)-graphs are Hamiltonian.

$R_r(3) = R(3, 3, \cdots, 3)$

- Much work on Schur numbers $s(r)$
  via sum-free partitions and cyclic colorings
  $s(r) > 89^{r/4 - c \log r} > 3.07^r$ [except small r]
  Abbott+ 1965+

- $s(r) + 2 \leq R_r(3)$
  $s(r) = 1, 4, 13, 44, \geq 160, \geq 536$

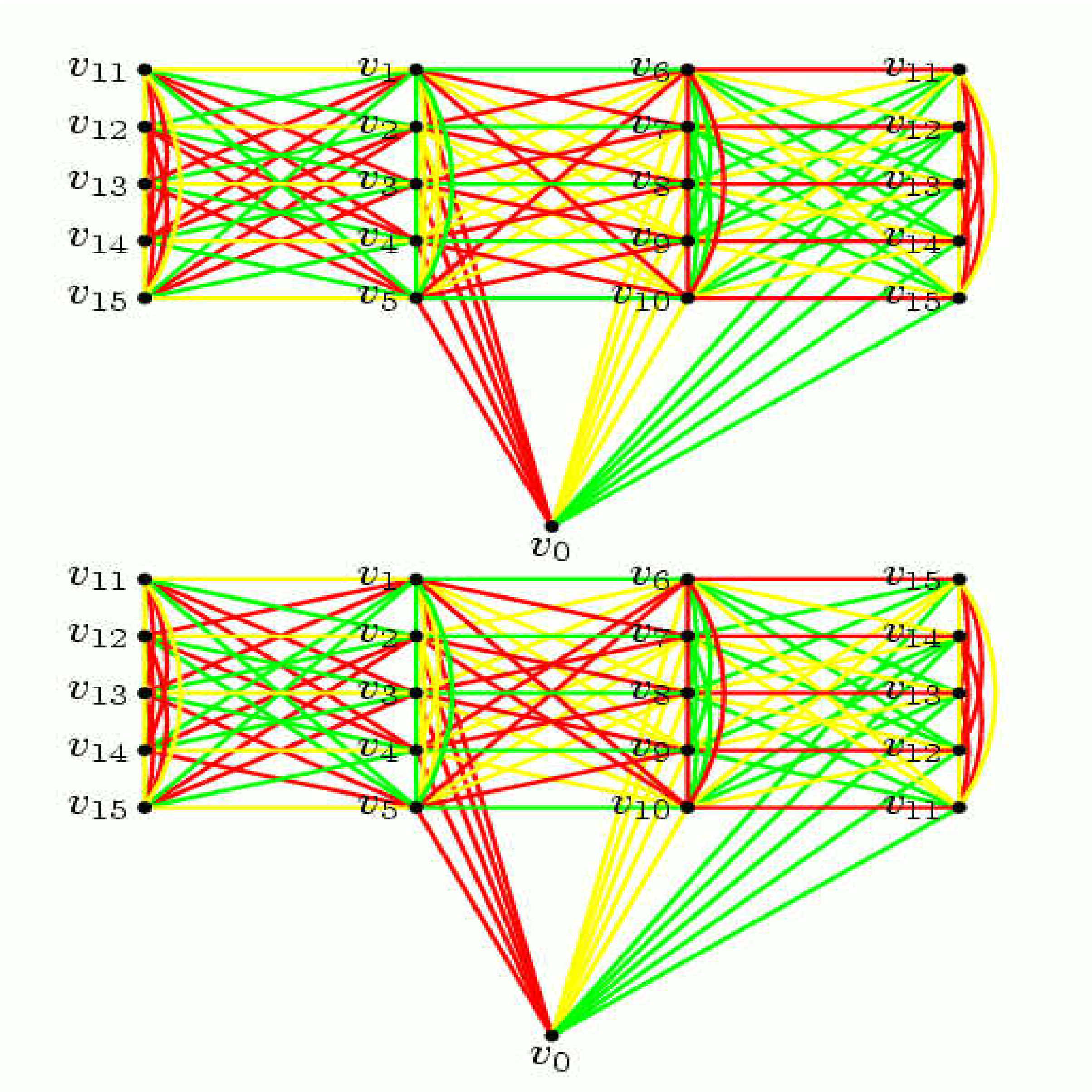
- $R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3$
  Chung 1973

- The limit $L = \lim_{r \to \infty} R_r(3)^{\frac{1}{r}}$ exists
  Chung-Grinstead 1983

  $(2s(r) + 1)^{\frac{1}{r}} = c_r \approx_{(r=6)} 3.199 < L$

$R(3,3,3) = 17$

two Kalbfleisch $(3,3,3;16)$-colorings, each color is a Clebsch graph



[Wikipedia]

Four colors - $R_4(3)$

$51 \leq R(3,3,3,3) \leq 62$

year	reference	lower	upper
1955	Greenwood, Gleason	42	66
1967	false rumors	[66]	
1971	Golomb, Baumert	46	
1973	Whitehead	50	65
1973	Chung, Porter	51	
1974	Folkman		65
1995	Sánchez-Flores		64
1995	Kramer (no computer)		62
2004	Fettes-Kramer-R (computer)		62

History of bounds on $R_4(3)$ [from FKR 2004]

$30 \leq R(3,3,4) \leq 31$

Theorem. Kalbfleisch 1965

$$30 \leq R(3,3,4)$$

Theorem. Piwakowski-R 2001
$R(3,3,4) = 31$ *if and only if there exists a* $(3,3,4;30)$*-coloring* C *such that every edge in the third color has at least one endpoint* x *with* $\deg_{C[3]}(x) = 13$*. Furthermore,* C *has at least* 25 *vertices* v *such that* $\deg_{C[1]}(v) = \deg_{C[2]}(v) = 8$ *and* $\deg_{C[3]}(v) = 13$*.*

$R(3,3,4)$

Note, March 2015.
Codish, Frank, Itzhakov, Miller, `arXiv` posting

BEE (Metodi-Codish 2013), Ben-Gurion University
Equi-propagation Encoder, a compiler to encode finite domain constraint problems to CNF.

Very significant progress of work towards the Ramsey number $R(3,3,4)$. Namely, they apply a SAT-solver to prove that if any $(3,3,4;30)$-coloring exists, then it must be 8-regular in the first two colors and 13-regular in the third.

Furthermore, they anticipate that full analysis of all such colorings will be completed, and thus the exact value of $R(3,3,4)$ will be known soon.

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$R(3,3,k)$

Theorem. Alon and Rödl 2005

$$R(3,3,k) = \Theta(k^3 \text{poly-log}\, k).$$

More general:
Avoid triangles in the first $r-1$ colors and K_k in color r,
then we have

$$R(3,\ldots,3,k) = \Theta(k^r \text{poly-log}\, k).$$

A nice, open, intriguing, feasible to solve case
(Exoo 1991, Piwakowski 1997)

$$28 \leq R_3(K_4 - e) \leq 30$$

What to do next?

theoretically and what to compute

Find new smart lower bound constructions

Explore relations between limits and Shannon capacity

Three colors

- improve $28 \leq R_3(K_4 - e) \leq 30$
- improve $45 \leq R(3,3,5) \leq 57$
- finish off $30 \leq R(3,3,4) \leq 31$

Four colors

- understand why heuristics don't find $51 \leq R_4(3)$
- improve on $R_4(3) \leq 62$

Papers to look at

- SPR, revision #14 of the survey paper
  Small Ramsey Numbers at the *EIJC*, January 2014.
- Xiaodong Xu and SPR,
  Some Open Questions for Ramsey and Folkman Numbers,
  Mittag-Leffler technical report, July 2014.
- Rujie Zhu, Xiaodong Xu, SPR,
  A step forwards on the Erdős-Sós problem concerning the
  Ramsey numbers $R(3, k)$, July 2015.

Thanks for listening!