Computation of the Ramsey Numbers $R(C_4, K_9)$ and $R(C_4, K_{10})$

Alexander Lange* Ivan Livinsky †

Stanisław Radziszowski ‡

Abstract

The Ramsey number $R(C_4, K_m)$ is the smallest n such that any graph on n vertices contains a cycle of length four or an independent set of order m. With the help of computer algorithms we obtain the exact values of the Ramsey numbers $R(C_4, K_9) = 30$ and $R(C_4, K_{10}) = 36$. New bounds for the next two open cases are also presented.

1 Introduction

Let G and H be simple graphs. An n-vertex graph F is a $(G, H; n)$-graph if it contains no subgraph isomorphic to G and $\overline{F}$ contains no subgraph isomorphic to H. Define $\mathcal{R}(G, H; n)$ to be the set of all such graphs. The Ramsey number $R(G, H)$ is the smallest n such that for every two-coloring of the edges of K_n, a monochromatic copy of G or H exists in the first or second color, respectively. Clearly, if a $(G, H; n)$-graph exists, then $R(G, H) > n$. It is known that Ramsey numbers exist [20] for all G and H. The values and bounds for various types of such numbers are collected and regularly updated by the third author [18].

The cycle-complete Ramsey numbers $R(C_n, K_m)$ have received much attention, both theoretically and computationally. For fixed $n = 3$, the problem becomes that of $R(3, k)$, which has been widely studied (see for example [24]), including the exact determination of its asymptotics [14].

*alange@uwaterloo.ca. Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON N2L 3G1.

†ivan.livinskyi@mail.utoronto.ca. Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4.

‡spr@cs.rit.edu. Department of Computer Science, Rochester Institute of Technology, Rochester, NY 14623.

Since 1976, it has been conjectured that $R(C_n, K_m) = (n-1)(m-1)+1$ for all $n \geq m \geq 3$, except $n = m = 3$ [10, 8]. Note that the lower bound is easy: $(m-1)$ vertex-disjoint copies of K_{n-1} provides a witness for $R(C_n, K_m) > (n-1)(m-1)$. For over 35 years, much work has been done to verify the upper bound, with $m = 8$ being the current smallest open case.

This work involves fixed $n = 4$, that is, the case of avoiding the quadrilateral C_4 in the first color. The currently best known asymptotic bounds for $R(C_4, K_m)$ are stated as Theorem 1.

Theorem 1 ([23, 2]). *There exist positive constants c_1 and c_2 such that*

$$c_1 \left(\frac{m}{\log m}\right)^{\frac{3}{2}} \leq R(C_4, K_m) \leq c_2 \left(\frac{m}{\log m}\right)^2.$$

The lower bound was obtained by Spencer in 1977 [23] using the probabilistic method. The upper bound was published by Caro, Li, Rousseau, and Zhang in 2000 [2], who in turn gave credit to an unpublished work by Szemerédi. The main challenge is determining whether $R(C_4, K_n) < n^{2-\epsilon}$ for some $\epsilon > 0$, a question posed by Erdős in 1981 [7].

Prior to this work, the exact values for $R(C_4, K_m)$ were known for $3 \leq m \leq 8$. Here, we present a computational proof that $R(C_4, K_9) = 30$ and $R(C_4, K_{10}) = 36$. The known values and bounds, including our new results, are gathered in Table 1.

m	$R(C_4, K_m)$	Year	References
3	7	1971	[3]
4	10	1972	[4]
5	14	1977	[5]
6	18	1987/1977	[9]/[21]
7	22	2002/1997	[19]/[12]
8	26	2002	[19]
9	30	this work	
10	36		
11	39–44		
12	42–52		

Table 1: Known values and bounds for $R(C_4, K_m)$.
Double references correspond to lower/upper bounds.

The value of $R(C_4, K_6)$ and bounds $21 \leq R(C_4, K_7) \leq 22$ were presented by Jayawardene and Rousseau in [12, 13]. The numbers $R(C_4, K_7)$, $R(C_4, K_8)$ and the bounds $30 \leq R(C_4, K_9) \leq 33$, $34 \leq R(C_4, K_{10}) \leq 40$

n	3	4	5	6	7	8	9	10	11	12
$\mathrm{ex}(n, C_4)$	3	4	6	7	9	11	13	16	18	21

n	13	14	15	16	17	18	19	20	21	22
$\mathrm{ex}(n, C_4)$	24	27	30	33	36	39	42	46	50	52

n	23	24	25	26	27	28	29	30	31	32
$\mathrm{ex}(n, C_4)$	56	59	63	67	71	76	80	85	90	92

Table 2: Known values for $\mathrm{ex}(n, C_4)$ [6, 27, 22].

were given by Radziszowski and Tse in [19]. Further upper bound improvements to 32 and 39 for $R(C_4, K_9)$ and $R(C_4, K_{10})$, respectively, were presented in [26].

For a graph G, $V(G)$ is the vertex set; $E(G)$ is the edge set; $N_G(v)$ is the neighborhood of $v \in V(G)$; $\deg_G(v)$ is $|N_G(v)|$; $\delta(G)$ is the minimum degree; and $\alpha(G)$ is the independence number.

2 Algorithms and Computations

2.1 Higher Level

The computations and algorithms used in this work are similar to those described in [19]. Comparable methods have been used to find other Ramsey numbers, such as in [17, 11].

The main idea behind the computations is to enumerate the sets $\mathcal{R}(C_4, K_m)$. If $\mathcal{R}(C_4, K_m; n) \neq \emptyset$, then $R(C_4, K_m) > n$, and if $\mathcal{R}(C_4, K_m; n+1) = \emptyset$, then $R(C_4, K_m) \leq n+1$. The latter is usually accomplished by extending $\mathcal{R}(C_4, K_m; t)$ to graphs in sets with higher m and/or t. Two methods used to achieve this are described in the next section.

Some special properties of C_4-free graphs proved useful during our computations. One such property involves an extremal Turán-type problem involving C_4-free graphs. Let $\mathrm{ex}(n, C_4)$ be the maximum number of edges of an n-vertex C_4-free graph. These numbers have been studied extensively both theoretically and computationally (cf. [1]). The values of $\mathrm{ex}(n, C_4)$ for $1 \leq n \leq 32$ are known [6, 27, 22] and they are displayed in Table 2. Two more useful properties are presented in Lemma 1.

Lemma 1 ([4, 1])**.** *If a C_4-free graph has n vertices, e edges, and minimum degree δ, then $\delta^2 - \delta + 1 \leq n$ and $e < \frac{1}{4}\, n\,(1 + \sqrt{4n-3})$.*

2.2 Methods

Our enumeration of various classes of (C_4, K_m)-graphs uses two computational methods, VertexExtend and Glue, described below.

VertexExtend

This algorithm extends a $(C_4, K_m; n)$-graph G to all possible $(C_4, K_m; n+1)$-graphs G' containing G by attaching a new vertex v to all feasible neighborhoods in G. By feasible, we mean that the additional edges do not create a C_4, while also preserving $\alpha(G') < m$. If complexity of computations is ignored, then full enumeration of $\mathcal{R}(C_4, K_m; n+1)$ can clearly be obtained from $\mathcal{R}(C_4, K_m; n)$ using this method.

Glue

The second method, called the Glue algorithm, constructs $\mathcal{R}(C_4, K_m; n+\delta+1)$ from $\mathcal{R}(C_4, K_{m-1}; n)$, where δ is the minimum degree of the new graphs. For a $(C_4, K_m; n+\delta+1)$-graph G, let v be a vertex of G such that $\deg_G(v) = \delta(G)$, and let X be the subgraph induced by $N_G(v)$; X must be a $(P_3, K_m; \delta)$-graph. Note that such a graph must be of the form $sK_2 \cup tK_1$, where $2s + t = \delta$ and $s + t < m$. Let Y be the induced subgraph of $V(G) \setminus (X \cup \{v\})$; Y must be a $(C_4, K_{m-1}; n)$-graph. If we know $\mathcal{R}(C_4, K_{m-1}; n)$, we can find all graphs in $\mathcal{R}(C_4, K_m; n+\delta+1)$ by considering how each vertex $x \in X$ can be connected to the vertices of Y. We call each neighborhood $N(x) \cap V(Y)$ the *cone* of x, denoted $c(x)$. We say that the cone $c(x)$ is *feasible* if:

1. $c(x)$ does not contain two endpoints of any P_3 in Y.
2. For distinct $x_1, x_2 \in V(X)$, $c(x_1) \cap c(x_2) = \emptyset$.
3. For each edge $\{x_1, x_2\} \in E(X)$, there is no $y_1 \in c(x_1)$ and $y_2 \in c(x_2)$ such that $\{y_1, y_2\} \in E(Y)$.
4. For each subgraph induced by $X' \subseteq X$ and Y' induced by $V(Y) \setminus \bigcup_{x \in X'} c(x)$, $\alpha(X') + \alpha(Y') < m$.

Conditions 1, 2, and 3 prevent C_4 subgraphs, while condition 4 prevents independent sets of order m. Figure 1 presents the main idea of Glue, while Figure 2 gives an explicit example of gluing a $(C_4, K_4; 9)$-graph to a $(C_4, K_5; 13)$-graph using $\delta = 3$.

2.3 Implementation and Optimization

Two separate implementations of VertexExtend and Glue were used in order to corroborate the correctness of the results. In all cases where both

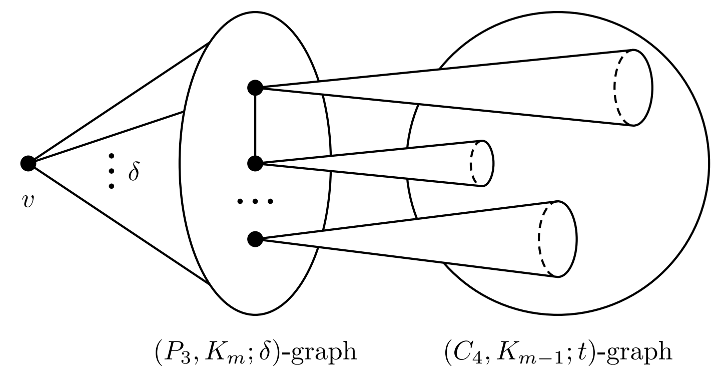

Figure 1: Gluing to a $(C_4, K_m; \delta + t + 1)$-graph.

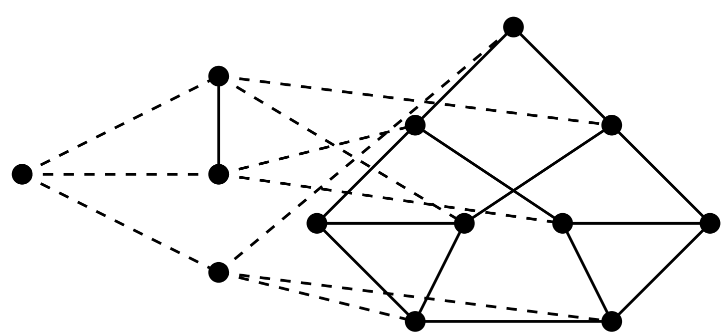

Figure 2: Gluing a $(C_4, K_4; 9)$-graph to a $(C_4, K_5; 13)$-graph.

implementations were used, the results agreed. We list the details of this agreement in the Appendix.

The rules for gluing (C_4, K_m)-graphs described in Section 2.2 allowed for a much needed speedup in computations. In most cases, it was beneficial to preprocess the Y graphs before gluing, storing information about the feasibility of the cones. For example, all subsets of vertices containing endpoints of a P_3 were removed from the list of feasible cones. Speed was greatly increased by precomputing the independence number $\alpha(Y')$ of each subgraph, which was critical for efficient testing of condition 4. This proved to be a bottleneck of the computations, and multiple strategies and implementations were tested. The most efficient algorithm implemented was based on *Algorithm 1: Precomputing independence number* of [11]. All data was stored in arrays of size 2^n, where the integer index of the array represented the bit-set of the vertices of the subgraph.

Two isomorphism testing tools were used in our implementations. The first implemented an algorithm described by William Kocay [15]. The other made use of the well-known software `nauty` by Brendan McKay [16].

3 Results

First, we obtained a full enumeration of $\mathcal{R}(C_4, K_7)$. This was significant, as the same enumeration was computationally infeasible when these methods were attempted in 2002 [19]. $\mathcal{R}(C_4, K_7)$ was first enumerated using VertexExtend. The same results were verified by gluing from $\mathcal{R}(C_4, K_6)$. For more information on these and similar consistency checks, see the Appendix. The statistics of $\mathcal{R}(C_4, K_7)$ by vertex and edge counts are displayed in Tables 3 and 4. The cases of counts found in [19] agree with ours.

e \ n	7	8	9	10	11	12	13	14	15
1	1								
2	2	1							
3	5	4	1						
4	9	9	4	1					
5	18	20	14	4	1				
6	29	42	40	16	3	1			
7	30	71	91	57	13	2			
8	17	88	178	172	56	9	1		
9	5	72	274	422	221	41	4		
10		31	289	805	737	183	19	1	
11		5	197	1135	1947	779	94	5	
12			74	1097	3861	2912	469	28	1
13			10	670	5405	8660	2221	151	5
14				222	5046	18943	9455	826	29
15				34	2965	28496	32805	4367	163
16				2	971	27902	84467	21211	920
17					146	16897	148686	87187	5218
18					11	5831	168441	277608	27740
19						1013	116266	622072	130043
20						82	45788	904916	507036
21						3	9434	801944	1513611
22							916	406222	3119854
23							39	108749	4033237
24							2	14039	3021620
25								818	1215627
26								24	241075
27								1	21639
28									851
29									22
30									2
Total	116	343	1172	4637	21383	111754	619107	3250169	13838693

Table 3: Statistics for $\mathcal{R}(C_4, K_7; n)$, $7 \leq n \leq 15$.
Note that for $n < 7$ the counts are for all C_4-free graphs.

e \ n	16	17	18	19	20	21
14	1					
15	5					
16	23	1				
17	116	3				
18	644	11	1			
19	3602	51	1			
20	19588	251	3			
21	97521	1311	12			
22	423964	6805	45			
23	1543985	33476	198			
24	4434855	149441	908			
25	9068568	585687	4045			
26	11612126	1964782	16971			
27	8299450	5448131	64462			
28	3016205	11583843	219831			
29	511367	16465694	672324	1		
30	37318	13277929	1813931	18		
31	1167	5287770	4096321	233		
32	26	938464	6953952	2399		
33	2	68369	7533349	17474		
34		2018	4275886	83786		
35		35	1064229	261093		
36		1	102512	520551		
37			3512	605219	1	
38			53	328849	12	
39			1	64919	126	
40				4132	999	
41				107	3611	
42				4	3762	
43					897	
44					53	
45					2	1
46						2
Total	39070533	55814073	26822547	1888785	9463	3

Table 4: Statistics for $\mathcal{R}(C_4, K_7; n)$, $16 \leq n \leq 21$.

e \ δ	1	2	3	4	Total
40		1			1
41		13			13
42		201			201
43		3055	108		3163
44		36884	8517		45401
45		302179	260678		562857
46	1	1449548	3502385	83	4952017
47	6	3662039	23059729	35368	26757142
48	29	4576213	75076644	1563123	81216009
49	53	2716695	110589375	11348103	124654226
50	27	744258	66302337	19535975	86582597
51	3	95358	15327155	9727032	25149548
52		5827	1352590	1588719	2947136
53		164	47152	94684	142000
54		6	732	2404	3142
55			4	37	41
56				1	1
Total	119	13592441	295527406	43895529	353015495

Table 5: Size vs minimum degree of graphs in $\mathcal{R}(C_4, K_8; 23)$. All such graphs with $\delta = 4$ were used with Glue to find $(C_4, K_9; 29)$-graphs.

Once $\mathcal{R}(C_4, K_7)$ was enumerated, we were able to construct $\mathcal{R}(C_4, K_8; n)$ for n equal to 23, 24, and 25. The gluing of $\mathcal{R}(C_4, K_8; 23)$ turned out to be the most computationally expensive, as there are 353015495 such graphs, but this was needed in order to extend them further to $\mathcal{R}(C_4, K_9; 29)$. The counts for $\mathcal{R}(C_4, K_8; 23)$ are displayed by size and minimum degree in Table 5. Statistics for $\mathcal{R}(C_4, K_8; 24)$ and $\mathcal{R}(C_4, K_8; 25)$ are gathered in Table 6. Our computations found that no (C_4, K_8)-graph exists with minimum degree 5.

3.1 $R(C_4, K_9)$

We constructed the sets $\mathcal{R}(C_4, K_9; 29)$ and $\mathcal{R}(C_4, K_9; 30)$ with the Glue algorithm. Since $R(C_4, K_8) = 26$, any $(C_4, K_9; 29)$-graph has minimum degree 3, 4, or 5 and can be obtained from $\mathcal{R}(C_4, K_8; n)$ for $n = 25, 24, 23$ by Glue. Note that the minimum degree of a $(C_4, K_8; 23)$-graph must be 4 in order to glue to a graph of minimum degree 5. This restriction improved the speed of computation, as there is a large number of $(C_4, K_8; 23)$-graphs

e \ n	24	25
48	1	
49	6	
50	48	
51	394	
52	3133	
53	21116	
54	60646	
55	57944	
56	18863	
57	2102	
58	96	2
59	4	10
60		15
61		9
Total	164353	36

Table 6: Statistics for $\mathcal{R}(C_4, K_8; n)$, $n = 24, 25$.
These graphs were used to find $(C_4, K_9; m)$-graphs for $m \geq 29$.

e \ δ	3	4	5	Total
70	1	1		2
71	8	5		13
72	12	11		23
73	18	33	1	52
74	10	64	7	81
75		49	9	58
76		19	7	26
77		6	4	10
78			2	2
Total	49	188	30	267

Table 7: Size vs minimum degree of graphs in $\mathcal{R}(C_4, K_9; 29)$.
These graphs were used to show that no $(C_4, K_{10}; 36)$-graph exists.

to consider. Statistics for $\mathcal{R}(C_4, K_9; 29)$ are found in Table 7.

Similarly, any $(C_4, K_9; 30)$-graph has minimum degree 4 or 5, and can be obtained from $\mathcal{R}(C_4, K_8; 25)$ or $\mathcal{R}(C_4, K_8; 24)$, respectively, via Glue . No $(C_4, K_9; 30)$-graphs were found, resulting in the following theorem.

Theorem 2. *$R(C_4, K_9) = 30$.*

3.2 $R(C_4, K_{10})$

Theorem 3. *$R(C_4, K_{10}) = 36$.*

Proof. We have found two 6-regular $(C_4, K_{10}; 35)$-graphs H_1 and H_2, establishing the lower bound. The orbits of H_1 are depicted in Figure 3 and its adjacency matrix is presented in Figure 4.

In order to prove $R(C_4, K_{10}) \leq 36$, it is necessary to show that no $(C_4, K_{10}; 36)$-graph exists. As $R(C_4, K_9) = 30$, from Lemma 1, we know that a $(C_4, K_{10}; 36)$-graph has minimum degree at most 6 and can be obtained from gluing a $(C_4, K_9; 29)$-graph. Gluing all of $\mathcal{R}(C_4, K_9; 29)$ resulted in finding no such graphs. □

The automorphism group $\mathrm{Aut}(H_1)$ has order 24 and its action on $V(H_1)$ has four orbits of 24, 6, 4, and 1 vertices, respectively. The automorphism group $\mathrm{Aut}(H_2)$ has order 40 and its action on $V(H_2)$ has three orbits of 20, 10, and 5 vertices. Both graphs H_1 and H_2 have 105 edges and 35 triangles. Each vertex is on three triangles, that is, each neighborhood is the union of three K_2 graphs. Both graphs are also bicritical: removing any edge produces an independent set of order 10, and adding any edge produces a C_4.

Interestingly, no $(C_4, K_{10}; n)$-graphs for $n = 34, 35$ were found by gluing from $\mathcal{R}(C_4, K_9; 29)$.

3.3 Higher Parameters

Theorem 4. *$39 \leq R(C_4, K_{11}) \leq 44$.*

Proof. The lower bound is obtained by construction. A $(C_4, K_{11}; 38)$-graph can easily be obtained by adding a triangle to H_1 or H_2.

If a $(C_4, K_{11}; 44)$-graph G exists, then from Lemma 1 it follows that G must have minimum degree at most 7. Such a graph can be obtained by applying Glue to a $(C_4, K_{10}; 36)$-graph. However, since $R(C_4, K_{10}) = 36$, no such graph exists, and therefore G does not exist as well. □

Theorem 5. *$42 \leq R(C_4, K_{12}) \leq 52$.*

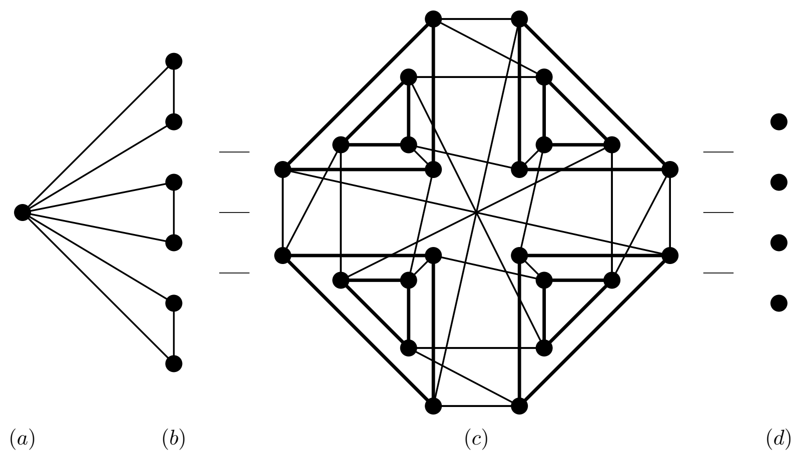

Figure 3: The four orbits of $\mathrm{Aut}(H_1)$. Parts (b) and (c) are connected by 24 edges, as well as (c) and (d).

Proof. The lower bound is obtained similarly as before: by adding a triangle to the $(C_4, K_{11}; 38)$-graphs of Theorem 4.

As $R(C_4, K_{11}) \leq 44$, any (C_4, K_{12})-graph can be obtained by applying Glue to a (C_4, K_{11})-graph with order at most 43. From Lemma 1, such a graph must have a minimum degree of at most 7, and therefore an order of at most 51. Thus, $R(C_4, K_{12}) \leq 52$. □

4 Acknowledgements

This research was done using resources provided by the Open Science Grid, which is supported by the National Science Foundation and the U.S. Department of Energy's Office of Science. We owe many thanks to Mats Rynge for his guidance throughout our use of the OSG, and to Gurcharan Khanna and Research Computing at RIT for their valuable and helpful support.

	a	b						c																								d			
a	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b	1	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
	1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
c	0	1	0	0	0	0	0	0	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0
	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1
	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
	0	0	0	1	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0
	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	1	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
	0	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0
	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	1	0
	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0
	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0	0	1
	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0
	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0
	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1
	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0
	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	1	0	0	1	0	0
d	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	1	0	0	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0

Figure 4: Adjacency matrix of H_1,
separated by the orbits of $\mathrm{Aut}(H_1)$.

References

[1] B. Bollobás. *Extremal Graph Theory.* Dover Publications, New York, 2004.

[2] Y. Caro, Y. Li, C. C. Rousseau, and Y. Zhang. Asymptotic bounds for some bipartite graph–complete graph Ramsey numbers. *Discrete Mathematics.* **220** (2000) 51–56.

[3] G. Chartrand and S. Schuster. On The Existence of Specified Cycles in Complementary Graphs. *Bulletin of the American Mathematical Society.* **77** (1971) 995–998.

[4] V. Chvátal and F. Harary. Generalized Ramsey Theory for Graphs, III. Small Off-Diagonal Numbers. *Pacific Journal of Mathematics.* **41** (1972) 335–345.

[5] M. Clancy. Some Small Ramsey Numbers. *Journal of Graph Theory.* **1** (1977) 89–91.

[6] C. Clapham, A. Flockhart, and J. Sheehan. Graphs Without Four-Cycles. *Journal of Graph Theory.* **13** (1989) 29–47.

[7] P. Erdős. On The Combinatorial Problems Which I Would Most Like To See Solved. *Combinatorica.* **1** (1981) 25–42.

[8] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp. On Cycle-Complete Graph Ramsey Numbers. *Journal of Graph Theory.* **2** (1978) 53–64.

[9] G. Exoo. Constructing Ramsey Graphs with a Computer. *Congressus Numerantium.* **59** (1987) 31–36.

[10] R. J. Faudree and R. H. Schelp. Some Problems in Ramsey Theory. In Y. Alavi and D. Lick, editors, *Theory and Applications of Graphs*, volume 642 of *Lecture Notes in Mathematics*, 500–515. Springer Berlin Heidelberg, 1978.

[11] J. Goedgebeur and S. P. Radziszowski. New Computational Upper Bounds for Ramsey Numbers $R(3,k)$. *The Electronic Journal of Combinatorics.* **20** (2013) 1–28.

[12] C. J. Jayawardene and C. C. Rousseau. An Upper Bound for the Ramsey Number of a Quadrilateral versus a Complete Graph on Seven Vertices. *Congressus Numerantium.* **130** (1998) 175–188.

[13] C. J. Jayawardene and C. C. Rousseau. The Ramsey Numbers for a Quadrilateral versus All Graphs on Six Vertices. *Journal of Combinatorial Mathematics and Combinatorial Computing.* **35** (2000) 71–87.

[14] J. H. Kim. The Ramsey Number $R(3,t)$ has Order of Magnitude $t^2/\log t$. *Random Structures and Algorithms.* **7** (1995) 173–207.

[15] W. Kocay. On Writing Isomorphism Programs. In W. Wallis, editor, *Computational and Constructive Design Theory*, 135–175. Kluwer Academic Publishers, 1996.

[16] B. McKay. nauty user's guide (version 2.5). *Australian National University, Department of Computer Science*. Software available at `http://cs.anu.edu.au/~bdm/nauty/`.

[17] B. D. McKay and S. P. Radziszowski. $R(4,5) = 25$. *Journal of Graph Theory.* **19** (1995) 309–322.

[18] S. P. Radziszowski. Small Ramsey Numbers. *Electronic Journal of Combinatorics.* (2014). DS 1, Revision #14. `http://www.combinatorics.org`.

[19] S. P. Radziszowski and K.-K. Tse. A Computational Approach for the Ramsey Numbers $R(C_4, K_n)$. *Journal of Combinatorial Mathematics and Combinatorial Computing.* **42** (2002) 195–207.

[20] F. P. Ramsey. On a Problem of Formal Logic. *Proceedings of London Mathematical Society.* **s2-30** (1930) 264–268.

[21] C. C. Rousseau and C. J. Jayawardene. The Ramsey Number for a Quadrilateral vs. a Complete Graph on Six Vertices. *Congressus Numerantium.* **123** (1997) 97–108.

[22] Z. Shao, J. Xu, and X. Xu. A New Turán Number for Quadrilateral. *Utilitas Mathematica.* **79** (2009) 51–58.

[23] J. Spencer. Asymptotic Lower Bounds for Ramsey Functions. *Discrete Mathematics.* **20** (1977) 69–76.

[24] J. Spencer. Eighty Years of Ramsey $r(3, k)$ ... and Counting! In A. Soifer, editor, *Ramsey Theory: Yesterday, Today and Tomorrow*, volume 285 of *Progress in Mathematics*, 27–39. Springer-Birkhauser, 2011.

[25] W. Stein et al. *Sage Mathematics Software (Version 3.5).* The Sage Development Team, 2012. `http://www.sagemath.org`.

[26] X. Xu, Z. Shao, and S. P. Radziszowski. Bounds on Some Ramsey Numbers Involving Quadrilateral. *Ars Combinatoria.* **90** (2009) 337–344.

[27] Y. Yuansheng and P. Rowlinson. On Extremal Graphs Without Four-Cycles. *Utilitas Mathematica.* **41** (1992) 204–210.

Appendix: Correctness of Computations

As our main results relied on the use of algorithms, it was important to take extra steps to verify the correctness of our implementations. Tests similar to those described in [17, 11] and others were, which are listed as follows:

1. Two independent implementations of Glue and VertexExtend were developed by the authors. The data was generated independently by both implementations, and all results agreed. The only data that was not produced by both was the full enumeration of $\mathcal{R}(C_4, K_8; 23)$ and $\mathcal{R}(C_4, K_9; 29)$, as this required the most computational resources. However, a partial set of $\mathcal{R}(C_4, K_8; 23)$, namely when $\delta = 1, 2$, was verified.

2. Both implementations were used to generate all graphs in $\mathcal{R}(C_4, K_t)$ for $4 \leq t \leq 7$. The results agreed, and gave the same counts as those found in [19].

3. For every $(C_4, K_8; 23)$-graph, we removed an edge if it did not increase the independence number to 8, therefore producing a different $(C_4, K_8; 23)$-graph. Every graph found this way was already included in the original set. For example, when going from size 51 to 50, 65059062 of the 86582597 graphs ($\approx 75\%$) were produced, none of which were new.

4. For every (C_4, K_8)-graph with 24 and 25 vertices, every vertex was removed, creating a (C_4, K_8)-graph with 23 and 24 vertices, respectively. Every graph produced was already included in the set obtained earlier.

5. Tests 3 and 4 were performed on other sets of graphs, including $\mathcal{R}(C_4, K_9; 29)$. Like before, all graphs obtained this way had already been found.

6. We extended $\mathcal{R}(C_4, K_9; 29)$ to $\mathcal{R}(C_4, K_9; 30)$ via VertexExtend and also obtained $\mathcal{R}(C_4, K_9; 30) = \emptyset$. Likewise, we extended the $(C_4, K_{10}; 35)$-graphs H_1 and H_2 from Theorem 3 and no $(C_4, K_{10}; 36)$-graphs were found.

7. All (C_4, K_m)-graphs were independently verified to not contain a C_4 or independent set of order m using the software `sage` [25].

Most of the large-scale computations were performed on the Open Science Grid. Over 175000 CPU hours (20 years) were used for these computations.