On Some Three-Color Ramsey Numbers for Paths

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Abstract

For graphs G_1, G_2, G_3, the three-color Ramsey number $R(G_1, G_2, G_3)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with 3 colors, then it contains a monochromatic copy of G_i in color i, for some $1 \leq i \leq 3$.

First, we prove that the conjectured equality $R(C_{2n}, C_{2n}, C_{2n}) = 4n$, if true, implies that $R(P_{2n+1}, P_{2n+1}, P_{2n+1}) = 4n + 1$ for all $n \geq 3$. We also obtain two new exact values $R(P_8, P_8, P_8) = 14$ and $R(P_9, P_9, P_9) = 17$, furthermore we do so without help of computer algorithms. Our results agree with a formula $R(P_n, P_n, P_n) = 2n-2+(n \bmod 2)$ which was proved for sufficiently large n by Gyárfás, Ruszinkó, Sárközy, and Szemerédi in 2007. This provides more evidence for the conjecture that the latter holds for all $n \geq 1$.

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1 Definitions

In this paper all graphs are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. For any edge coloring F of a complete graph, F^i will denote the graph induced by the edges of color i in F. Let P_k (resp. C_k) be the path (resp. cycle) on k vertices. The *circumference* $c(G)$ of a graph G is the length of its longest cycle.

The Turán number $T(n,G)$ is the maximum number of edges in any n-vertex graph that does not contain any subgraph isomorphic to G. A graph on n vertices is said to be *extremal with respect to* G if it does not contain a subgraph isomorphic to G and has exactly $T(n,G)$ edges.

For given graphs $G_1, G_2, ..., G_k, k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, ..., G_k)$ is the smallest integer n such that if we arbitrarily color with k colors the edges of the complete graph of order n, K_n, then it contains a monochromatic copy of G_i in color i, for some $1 \leq i \leq k$. A coloring of the edges of K_n with k colors is called a $(G_1, G_2, ..., G_k; n)$-coloring, if it does not contain a subgraph isomorphic to G_i in color i for any $1 \leq i \leq k$. In the diagonal cases, when for all i, $1 \leq i \leq k$, $G_i = G$ for some graph G, we will write $R(G_1, G_2, ..., G_k) = R_k(G)$. Finally, we will refer to the first three colors of such Ramsey colorings as red, blue and green, respectively.

2 Overview

In this article we study the values of three-color diagonal Ramsey numbers for paths. In the case of two color Ramsey numbers, a well known theorem of Gerencsér and Gyárfás [7] states that $R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ for $n \geq m \geq 2$.

Clearly, we have $R_3(P_1) = 1$ and $R_3(P_2) = 2$. The cases $R_3(P_3) = 5$ and $R_3(P_4) = 6$ are easy but need some thought, while the results $R_3(P_5) = 9$, $R_3(P_6) = 10$ and $R_3(P_7) = 13$ already required help of computer algorithms (see section 6.4.1 of [13] for details and references to these and other related cases). The first open cases are those of $R_3(P_8)$ and $R_3(P_9)$, which are determined later in this paper. All known values agree with a very remarkable result obtained by Gyárfás, Ruszinkó, Sárközy, and Szemerédi in 2007 [10] formulated as follows.

Theorem 1 ([10]) *For all sufficiently large n, we have*

$$R_3(P_n) = \begin{cases} 2n-1 & \text{for odd } n, \\ 2n-2 & \text{for even } n. \end{cases} \tag{1}$$

The proof of Theorem 1 is very long and complicated. Our attempts to extract from it any reasonable bound on how large n should be for (1) to hold, failed. Actually, Faudree and Schelp [6], already in 1975, stated that "they feel" that (1) holds for all n. They did so when considering more general cases of $R(P_m, P_n, P_k)$ for paths of different lengths. We believe that the diagonal case deserves the status of a conjecture.

Conjecture 2 ([6]) $R_3(P_n) = 2n - 2 + (n \bmod 2)$ *holds for all* $n \geq 1$.

Let $N = 2n - 3 + (n \bmod 2)$. It is known that $R_3(P_n) > N$ since one can find a $(P_n, P_n, P_n; N)$-coloring for all $n \geq 1$. For $n = 1, 2$, these are on an empty and one-element set of vertices, respectively. For $n = 3$ and $N = 4$, the partition of the edges of K_4 into 3 matchings $2K_2$ in 3 distinct colors gives a witness coloring for $R_3(P_3) > 4$. For the general case, one can obtain a $(P_n, P_n, P_n; N)$-coloring by using a "blow-up" of this factorization of K_4 as follows. For odd $n = 2m-1 \geq 5$, a witness coloring for $R_3(P_n) > N = 4m-4$ can be obtained by blowing up each vertex of such colored K_4 into a set of $m - 1$ vertices, and coloring the edges within the new 4 sets arbitrarily. Similarly for $n = 2m$, a witness coloring for $R_3(P_n) > 4m-3$ can be obtained by blowing up three vertices of K_4 to $m - 1$ vertices, and one to m vertices (for more details see [10]).

It is interesting to compare (1) to the conjectured values of three-color diagonal Ramsey numbers for cycles.

Conjecture 3 ([4][3])

$$R_3(C_n) = \begin{cases} 4n-3 & \text{for odd } n \geq 5, \\ 2n & \text{for even } n \geq 6. \end{cases} \tag{2}$$

The odd case was conjectured by Bondy and Erdős in 1981 [4], while the even case by the second author in 2005 [3]. Like with (1) for paths, (2) is known to hold for all sufficiently large n. For the odd n odd case, this result and an outline of the proof was described by Kohayakawa, Simonovits and

Skokan in 2005 [11], and the full proof by the same authors is to appear [12]. The case for even n was settled by Benevides and Skokan in 2009 [1]. These results followed the exact asymptotic results obtained by Łuczak and others (see also section 6.3.1 of [13] for details and references to other related cases). We know that (2) holds for all $n \geq n_0$ for some n_0, though there seems to be no easy way to find any concrete upper bound on n_0. The first open cases of Conjecture 3 are those of $R_3(C_9)$ and $R_3(C_{10})$.

In section 4 we will prove an interesting implication that the even n case of (2) implies the odd $(n+1)$ case of (1) for $n \geq 6$. The equalities $R_3(C_6) = 12$ [17] and $R_3(C_8) = 16$ [14] were obtained with the help of computer algorithms. Thus, it will imply that $R_3(P_7) = 13$ and $R_3(P_9) = 17$. We will also provide a computer-free proof of the latter. Finally, we prove that $R_3(P_8) = 14$, which leaves $R_3(P_{10})$ as the first open case of (1).

3 Background Results

Gyárfás, Rousseau and Schelp [9] completely solved the question of what is the maximum number of edges $g(m, n, k)$ in any P_k-free subgraph of the complete bipartite graph $K_{m,n}$. They also characterized all the corresponding extremal graphs. Tables III and IV in [9] present formulas for $g(m, n, k)$ for even and odd k, respectively, and Tables I and II therein describe the constructions of the extremal graphs achieving $g(m, n, k)$. In our proofs of sections 4 and 5 we will refer to these tables several times, and hence they are reproduced in Appendix A (with some additional comments) for convenience of the readers.

Also in the proofs we will need some values of Turán numbers for paths. In order to determine the required $T(n, P_k)$, the following theorem by Faudree and Schelp [6], which enhances and condenses the results by Erdős and Gallai [5], will be used. For vertex-disjoint graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, the join $H_1 + H_2$ is a graph on vertices $V_1 \cup V_2$ with the set of edges equal to $E_1 \cup E_2 \cup \{\{u, v\} \mid u \in V_1, v \in V_2\}$.

Theorem 4 ([6][5]) *If G is a graph with $|V(G)| = kt + r$, $r < k$, $0 \leq t, r$, containing no P_{k+1}, then $|E(G)| \leq t\binom{k}{2} + \binom{r}{2}$ with equality if and only if G is either $(tK_k) \cup K_r$ or $((t-l-1)K_k) \cup (K_{(k-1)/2} + \overline{K}_{(k+1)/2+lk+r})$ for some $0 \leq l < t$ when k is odd, $t > 0$, and $r = (k \pm 1)/2$.*

The following notation and terminology comes from [2]. For positive integers a and b, define $r(a,b)$ as

$$r(a,b) = a - b\left\lfloor \frac{a}{b} \right\rfloor = a \bmod b.$$

For integers $n \geq k \geq 3$, define $w(n,k)$ by

$$w(n,k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1), \tag{3}$$

where $r = r(n-1, k-1)$.

Woodall's theorem [15] can then be formulated as follows.

Theorem 5 ([2]) *Let G be a graph on n vertices and m edges with $m \geq n$ and circumference $c(G)$ equal to k. Then*

$$m \leq w(n,k),$$

and this result is best possible.

In [2] (see also Appendix B), one can find the description of all extremal graphs achieving $w(n,k)$.

4 Progress on $R_3(P_{2n+1})$

First we prove the following general implication.

Theorem 6 *For all $n \geq 3$, if $R_3(C_{2n}) = 4n$, then $R_3(P_{2n+1}) = 4n+1$.*

Proof. The lower bound follows from the "blow-up" construction commented on after the statement of Conjecture 2 in section 2 (see also [10]).

For the upper bound, suppose that there exists a 3-edge coloring of K_{4n+1} without monochromatic P_{2n+1}. From the assumption that $R_3(C_{2n}) = 4n$, we know that this coloring contains a monochromatic C_{2n} (actually, a weaker assumption $R_3(C_{2n}) \leq 4n+1$ would suffice, but in Theorem 6 we preferred to stress directly a link between Conjectures 2 and 3). Without loss of generality, we assume that this C_{2n} is red. Now, in order to avoid red P_{2n+1} no vertex on this cycle can be connected by a red edge to any vertex outside of the cycle. Hence, we have a complete bipartite graph $K_{2n,2n+1}$ with only blue and green edges. Let the parts of this bipartite graph be called X (vertices on the cycle) and Y (vertices outside of the cycle). Using notation of [9], we have

$$a = 2n = |X|, \quad b = 2n+1 = |Y|, \quad c = n-1, \quad a = 2(c+1),$$

hence, if we apply the last row of Table IV (see [9] or Appendix A), then we obtain that the maximum number of edges $g(2n, 2n+1, 2n+1)$ in any P_{2n+1}-free subgraph of $K_{2n,2n+1}$ is

$$g(2n, 2n+1, 2n+1) = (a+b-2c)c = 2n^2 + n - 3.$$

This implies that

$$2g(2n, 2n+1, 2n+1) = 4n^2 + 2n - 6 < 2n(2n+1) = |X||Y|,$$

and therefore blue and green edges cannot account for all the edges of $K_{X,Y}$ without creating a monochromatic P_{2n+1}. This completes the proof of the upper bound $R_3(P_{2n+1}) \leq 4n+1$. $\square$

Corollary 7 (*implied by computations for cycles*)

$$R_3(P_7) = 13 \quad \textit{and} \quad R_3(P_9) = 17.$$

Proof. The equality $R_3(P_7) = 13$ was obtained previously by direct computations [16], while the result $R_3(P_9) = 17$ is new. It is known that $R_3(C_6) = 12$ [17] and $R_3(C_8) = 16$ [14], where the upper bounds in both equalities were obtained with the help of computer algorithms. By Theorem 6, these imply that $R_3(P_7) = 13$ and $R_3(P_9) = 17$. $\square$

In the proof of the next theorem we provide a computer-free proof of the upper bound $R_3(P_9) \leq 17$. The proof of $R_3(P_7) \leq 13$ can be obtained by a similar reasoning.

Theorem 8 (*computer-free*)

$$R_3(P_9) \leq 17.$$

Proof. We need to show that each 3-coloring of the edges of K_{17} contains a monochromatic P_9. Let us suppose that there is a $(P_9, P_9, P_9; 17)$-coloring G with colors red, blue and green, forming graphs G^1, G^2 and G^3, respectively. Since K_{17} has 136 edges, we may assume without loss of generality that there are at least 46 red edges, i.e. $e(G^1) \geq 46$.

Since by (3) we have $w(17, 6) = 46$, it follows by Theorem 5 that G^1 contains a cycle C_k for some $k \geq 6$. One can easily verify that the critical graphs in this case (described in [2] and in Appendix B) have P_9, and thus $k \geq 7$. If $k \geq 9$, we immediately obtain a red P_9, a contradiction. If $k = 8$, then we use an argument very similar to the main steps in the proof of Theorem 6. In order to avoid a P_9 in G^1 we have a bipartite graph with partite sets of order 8 and 9, respectively, and let us denote it by G'. In order to avoid monochromatic P_9 in G^2 and G^3, by using the last row of Table IV, we see that G' can contain at most 66 blue and green edges. Each of at least 6 other edges of G' are red and together with the C_8 they contain a red P_9, again a contradiction. Hence, in the rest of the proof we will assume that G has a red C_7 with vertices $C = \{c_1, c_2, ..., c_7\}$, the remaining vertices are $P = \{p_1, p_2, ..., p_{10}\}$, and G has no red C_k for any $k \geq 8$.

Claim 8A. *Let H be a $(P_9, P_9, P_9; 17)$-coloring, and suppose that $|H^1| \geq 46$, $c(H^1) = 7$, and H^1 contains a cycle $C = C_7$. Then there are at least 4 vertices in $V(H) \setminus V(C)$ joined by at least one red edge to C.*

Proof of Claim 8A. Consider the coloring H as stated above. Let the vertices of C_7 in H^1 be $C = \{c_1, c_2, ..., c_7\}$, and the remaining vertices of H are $P = \{p_1, p_2, ..., p_{10}\}$. We will use the tables in Appendix A several times when considering bipartite subgraphs of $K_{|C|,k} = K_{7,k}$ for $k = 10, 9, 8$ and 7. We prove that there are red edges in these bipartite subgraphs, and so for each k we obtain one more vertex in $V(H) \setminus C$ joined to the cycle C_7 by at least one red edge.

The maximum possible number of edges in a bipartite graph with partite sets 7 and $k = 10$ or 9 without P_9 is $7 + 3(k - 1)$, which follows from Table IV with $a = 7$, $b = k$, $c = 3$ and $f_1(a, b, c) = a + (b - 1)c$. Since

$2(7+3(k-1)) < 7k = e(K_{7,k})$ for $k = 10$ and 9, we obtain the first 2 vertices, say p_1 and p_2, connected to C_7 by at least one red edge.

Now, we consider the bipartite graph $K_{|C|,|P\setminus\{p_1,p_2\}|} = K_{7,8}$. Similarly to the previous case, the maximum number of edges in this bipartite graph without P_9 is $7+3(8-1) = 28$. This time, however, this is exactly half of the edges of $K_{7,8}$, so we need to consider the possible extremal graphs. By Table II these extremal graphs are G_{14} and G_{15} with $a = 7$, $b = 8$ and $c = 3$, and they can be eliminated as follows:

- $G_{14} = K_{7,8} - K_{4,7}$. Clearly, $K_{4,7}$ contains a P_9, so it cannot consist of edges of a single color, a contradiction.

- $G_{15} = K_{4,4} \cup K_{3,4}$, which under bipartite complement is isomorphic to itself. Let us consider vertex p_1. To avoid red C_8 the vertex p_1 is joined by at most 3 red edges to the cycle C_7. By considering the remaining edges from the vertex p_1 we see that at least one, say blue, is connected to the $K_{4,4}$ part of G_{15} in blue. This easily gives a monochromatic P_9, a contradiction.

Thus, we have the third vertex, say p_3, connected to C by a red edge.

Now we consider the bipartite graph $K_{C,P\setminus\{p_1,p_2,p_3\}}$. By the third case in Table IV, the maximum possible number of edges in this bipartite graph without P_9 is $7+3(7-1) = 25$. By Table II, there are two possible extremal graphs: G_{14} and G_{15} which now are $K_{7,7} - K_{4,6}$ and $K_{4,4} \cup K_{3,3}$, respectively. We proceed similarly as for $k = 8$, now by considering possible edges from p_1, p_2 and p_3 to the cycle C_7. As for $k = 8$, we have to consider two cases.

- G_{14}. Suppose that $K_{7,7} - K_{4,6}$ is blue, and its bipartite complement $K_{4,6}$ is green. In order to avoid red C_8, each of p_1 and p_2 may have at most 3 red edges to C_7, and thus at most 3 to the left side of $K_{4,6}$. If any of the vertices p_1 and p_2 is connected by a green edge to $K_{4,6}$, then we have a green P_9. Otherwise, both p_1 and p_2 have a blue edge to $K_{4,6}$, which easily leads to a blue P_9.

- G_{15}. Suppose that $K_{4,4} \cup K_{3,3}$ is blue, and its bipartite complement $K_{4,3} \cup K_{3,4}$ is green. In order to avoid red C_8, each of p_1 and p_2 may have at most 3 red edges to C_7, and thus at most 3 to the left side L of $K_{4,4}$. If any of the edges from p_1 or p_2 to L is blue, then we have a blue P_9. Otherwise, each of p_1 and p_2 has at least one green edge to L;

say $\{p_1, x\}, \{p_2, y\}$ are green for some $x, y \in L$. If $x \neq y$ then we have a green P_9. Thus we can assume that for some $x \in L$ and for $i = 1, 2, 3$ we have: $\{p_i, x\}$ is green, the edges $\{p_i, z\}$ are red for $x \neq z \in L$, and all edges from p_i to the left side of $K_{3,3}$ are blue. Now, in order to avoid red C_8 and green P_9, all three edges between p_i's must be blue. This leads to a blue P_9.

Hence, we obtain the fourth required vertex p_4. This completes the proof of Claim 8A.

We have $m \geq 4$ vertices $M = \{p_1, \ldots, p_m\} \subset P$ not on the red C_7 joined to it by some red edges. Assuming that there is no red P_9, one can easily see that there can be no red edges $\{p_i, p_j\}$ with $p_i \in M$ and $p_j \in P$. Hence, P induces at most $\binom{10-m}{2}$ red edges. To avoid red C_8, the vertices in M can be joined by at most 3 red edges each to C (to vertices nonadjacent on the cycle C_7).

First, consider the case when a vertex in M, say p_1, has 3 red edges to C, without loss of generality $\{p_1, c_1\}$, $\{p_1, c_3\}$ and $\{p_1, c_5\}$. Note that no vertex $p \in M$, $p \neq p_1$, can be joined by any red edge to the vertices in the set $\{c_2, c_4, c_6, c_7\}$, since otherwise a red P_9 from p to p_1 can be easily constructed. In addition, if the edge $\{p_2, c_5\}$ is red, then the edges $\{c_4, c_6\}$, $\{c_4, c_7\}$, $\{c_2, c_4\}$ are blue or green. For example, if $\{c_4, c_6\}$ is red, then $p_2c_5c_4c_6c_7c_1c_2c_3p_1$ is a red P_9. Similarly, if $\{p_2, c_1\}$ or $\{p_2, c_3\}$ is red, then at least three edges induced in C must be blue or green. In all cases, C induces at most 18 red edges. Thus, counting red edges in C, between C and P, and in P, we have

$$e(G^1) \leq 18 + 3m + \binom{10-m}{2}. \tag{4}$$

Observe that the set $M \cup \{c_2, c_4, c_6, c_7\}$ induces only blue and green edges. The latter and the known value $R(P_9, P_9) = 12$ [7] imply that $m + 4 \leq 11$. By Claim 8A we have $m \geq 4$, so $4 \leq m \leq 7$, and we find that $e(G^1) < 46$ for all possible m. This is a contradiction.

Finally we consider the case when all vertices in M are connected to C by at most 2 red edges. Counting again red edges, for all possible $4 \leq m \leq 10$, we obtain

$$e(G^1) \leq \binom{7}{2} + 2m + \binom{10-m}{2} < 46, \tag{5}$$

which is a contradiction. $\square$

5 Ramsey Number $R_3(P_8)$

We begin with a lemma which is technically very similar to Claim 8A within the proof of Theorem 8.

Lemma 9 *Let H be a $(P_8, P_8, P_8; 14)$-coloring, and suppose that $|H^1| \geq 31$, $c(H^1) = 6$, and H^1 contains a cycle $C = C_6$. Then there are at least 3 vertices in $V(H) \setminus V(C)$ joined by at least one red edge to the cycle C.*

Proof. We prove this lemma similarly as Claim 8A in Theorem 8. Consider any coloring H as stated above. Let the vertices of C_6 in H^1 be $C = \{c_1, c_2, ..., c_6\}$, and the remaining vertices are $P = \{p_1, p_2, ..., p_8\}$.

We will use the tables of Appendix A several times when considering bipartite subgraphs of $K_{|C|,k} = K_{6,k}$ for $k = 8, 7$ and 6. We prove that there are red edges in these bipartite subgraphs, and so for each k we obtain one more vertex in $V(H) \setminus C$ joined to C by at least one red edge. By using three times Tables I and III and considering $K_{6,k}$ for $k = 8, 7, 6$, we can see that the maximum number of edges in these bipartite subgraphs without P_8 is $3k$. From Table I, the extremal graphs are $K_{3,l} \cup K_{3,k-l}$, where $0 \leq l \leq k$.

First, consider the case when $k = 8$. The maximum number of edges in $K_{6,8}$ without P_8 is 24. Without loss of generality consider the situation when $K_{3,l} \cup K_{3,8-l}$, $l \geq 4$, is blue and the bipartite complement of this graph is green. Assume that $K_{3,l} = K_{3,|S|}$ where $S = \{p_1, p_2, ..., p_l\}$. These graphs can be eliminated as follows:

- $l \geq 5$. To avoid a blue or green P_8, S has only red edges and the vertices in S can be joined only by red edges to vertices in $P \setminus S$. This easily gives a red P_8, a contradiction.
- $l = 4$. To avoid a blue or green P_8, all the edges from S to $P \setminus S$ are red. Then we have a red P_8, a contradiction.

Thus, we have the first vertex, say p_1, connected to C by a red edge.

Now, we consider the bipartite graph $K_{|C|,|V(P)-\{p_1\}|} = K_{6,7}$. Similarly as in the previous case, the maximum number of edges without P_8 is 21. Without loss of generality, consider the situation when $K_{3,l} \cup K_{3,7-l}$, $l \geq 4$, is blue and the bipartite complement of this graph is green. Assume that $K_{3,l} = K_{3,|S|}$. To avoid a red P_8, p_1 has only blue or green edges to the set S.

Then we have a blue or green P_8, a contradiction. Thus, we have the second vertex, say p_2, connected to C by a red edge.

Now, we consider the bipartite graph $K_{|C|,|V(P)-\{p_1,p_2\}|} = K_{6,6}$. Similarly as in the previous case, the maximum number of edges in this bipartite graph without P_8 is 18. Without loss of generality let us consider the situation when $K_{3,l} \cup K_{3,6-l}$, $l \geq 3$, is blue and the bipartite complement of this graph is green. Assume that $K_{3,l} = K_{3,|S|}$. To avoid a red P_8, the edge $\{p_1, p_2\}$ is blue or green and p_1, p_2 have only blue or green edges to the set S. Then we have a blue or green P_8, a contradiction. Hence, we obtain the third required vertex p_3, which completes the proof of Lemma 9. □

Theorem 10 *Three-color Ramsey number of the path P_8 satisfies*

$$R_3(P_8) = 14.$$

Proof. We need to show that every 3-edge coloring of K_{14} contains a monochromatic P_8. Let us suppose that there is a $(P_8, P_8, P_8; 14)$-coloring G with colors red, blue and green, forming graphs G^1, G^2 and G^3, respectively. Since K_{14} has 91 edges, we may assume without loss of generality that there are at least 31 red edges, i.e. $e(G^1) \geq 31$.

Since by (3) we have $w(14, 5) = 31$, it follows by Theorem 5 that G^1 contains a cycle C_k for some $k \geq 5$. One can routinely verify that the critical graphs in this case (described in [2] and Appendix B) have P_8, and thus $k \geq 6$. If $k \geq 8$, then we immediately obtain a P_8, a contradiction. If $k = 7$, then to avoid a P_8 in G^1 we have a bipartite graph G' with two partite sets of order 7. In order to avoid monochromatic P_8 in G^2 and G^3, by using row 3 in Table III, we see that the graph G' can contain at most 48 blue and green edges. At least one remaining edge of G' is red and together with the C_7 we have a red P_8, a contradiction. Hence, in the rest of the proof we will assume that G has a red C_6 with vertices $C = \{c_1, c_2, ..., c_6\}$, the remaining vertices are $P = \{p_1, p_2, ..., p_8\}$, and G has no red C_k for any $k \geq 7$.

By Lemma 9 we have $m \geq 3$ vertices $M = \{p_1, \ldots, p_m\} \subset P$ not on the red C_6, joined to it by some red edges. Assuming that there is no red P_8, one can easily see that there can be no red edges $\{p_i, p_j\}$ with $p_i \in M$ and $p_j \in P$. Hence P induces at most $\binom{8-m}{2}$ red edges. To avoid red C_7, the vertices in M can be joined by at most 3 red edges each to C (to vertices nonadjacent on the cycle C_6). We will be counting red edges in C, between C and P, and in P, similarly as in (4) and (5).

First, consider the case when all the vertices in M are connected to C by at most 2 red edges each. If at least one them is connected to 2 vertices in C, then at least one of the edges induced by C is not red, or there are less than $2m$ edges between C and P. Hence, for all possible $3 \leq m \leq 8$, we have

$$e(G^1) \leq \binom{6}{2} + 2m - 1 + \binom{8-m}{2} < 31,$$

which gives a contradiction.

The remaining case is when some vertex, say p_1, in M is connected to C by exactly 3 red edges, and the red edges from p_1 to C are $\{p_1, c_1\}$, $\{p_1, c_3\}$, $\{p_1, c_5\}$. Recall that now we can also assume that G has no red C_7. Then no vertex $p_i \in P$, $2 \leq i \leq 8$, can be joined by a red edge to any of the vertices in the set $\{c_2, c_4, c_6\}$. In addition, if the edge $\{p_2, c_1\}$ is red, then the edges $\{c_2, c_4\}$, $\{c_2, c_6\}$, $\{c_4, c_6\}$ are blue or green. For example, if $\{c_2, c_4\}$ is red, then $p_2c_1c_2c_4c_3p_1c_5c_6$ is a red P_8. Similarly, if $\{p_2, c_3\}$ or $\{p_2, c_5\}$ is red, then at least the same three edges induced in C must be blue or green. In all cases, C induces at most 12 red edges.

Observe that the set $M \cup \{c_2, c_4, c_6\}$ contains only blue and green edges. The latter and the known value $R(P_8, P_8) = 11$ [7] imply that $m + 3 \leq 10$. Note that if $m = 7$, then the sole vertex in $P \setminus M$ is also not in any red edge. Therefore, $P \cup \{c_2, c_4, c_6\}$ contains only blue and green edges and, again because of $R(P_8, P_8) = 11$, there must be a monochromatic P_8. Hence, we can assume that $3 \leq m \leq 6$. This time we obtain

$$e(G^1) \leq 12 + 3m + \binom{8-m}{2}. \tag{6}$$

Now $e(G^1)$ can achieve 31 in (6) for $m = 3$ and $m = 6$, furthermore only in cases when all (3 or 6) vertices in M are connected by exactly 3 red edges to C. We will show that in both cases G has a blue or green P_8.

If $m = 6$, then the equality in (6) implies that $P \cup \{c_2, c_4, c_6\}$ contains exactly one red edge in $P \setminus M$, or equivalently, the $K_{11} - e$ with vertices $P \cup \{c_2, c_4, c_6\}$ has all its 54 edges blue or green. By Theorem 4 with $k = 7$, $t = 1$ and $r = 4$ we obtain $T(11, P_8) = 27$. One can easily check that it is not possible for two copies of the corresponding extremal graphs to cover $K_{11} - e$.

The last situation to consider is that of $m = 3$, where G^1 has two components: one spanned by 9 vertices of $C \cup M$ with 21 red edges and a red K_5 on vertices $Q = P \setminus M = \{p_4, \ldots, p_8\}$. The set $H = M \cup \{c_2, c_4, c_6\}$ has no red edges. Denote by R the set $\{c_1, c_3, c_5\}$. The 60 edges of $G^2 \cup G^3$ form a complete K_6 on H and a complete bipartite graph $K_{Q,H \cup R}$. Let P_l be the longest monochromatic, say blue, path in H, and denote by x and y its endpoints. By Theorem 4 we have $T(6, P_4) = 6$, which implies that $l = 6$ or $l = 5$ (it is also implied by $R(P_5, P_5) = 6$). We have the following possibilities:

Case 1. There are no blue edges joining x or y to Q (for $l = 5$ or $l = 6$).

We have $H \cup R = C \cup M$, and let $S = C \cup M \setminus \{x, y\}$. We consider the complete bipartite graph $K_{5,7}$ with partite sets Q and S. Because all the edges from x and y to Q are green, this $K_{Q,S}$ cannot have green P_4. The third row of Table III with $a = 5$, $b = 7$ and $c = 1$, implies that there are at most 10 green edges between Q and S. Clearly, $K_{Q,S}$ cannot have blue P_8. We now use the second row of the same Table III with $c = 3$, and see that there are at most 21 blue edges between Q and S. There are not enough green and blue edges to cover all 35 edges of $K_{Q,S}$, which is a contradiction.

Case 2. There is a blue edge from x or y to Q, say $\{x, p_4\}$, and $l = 6$.

Let the blue P_l in H be $xs_1s_2s_3s_4y$. If there is no blue P_8, then all the edges joining y to p_i, $5 \leq i \leq 8$, and joining p_4 to R are green. We consider the colors of the edges from s_4 to the set $Q \setminus \{p_4\} = \{p_5, p_6, p_7, p_8\}$. This case is now broken into three subcases, as follows:

1. There are at least two blue edges from s_4 to $Q \setminus \{p_4\}$, say $\{s_4, p_5\}$ and $\{s_4, p_6\}$. To avoid blue P_8 all the edges between $\{p_5, p_6\}$ and R must be green, but in this case we have a green $P_8 = p_8yp_5c_1p_6c_3p_4c_5$.

2. There is exactly one such blue edge, say $\{s_4, p_5\}$. To avoid blue P_8 all the edges between p_5 and R must be green, but then we have a green $P_8 = c_1p_4c_3p_5yp_6s_4p_7$.

3. All edges from s_4 to $\{p_5, p_6, p_7, p_8\}$ are green. If there is a green edge between R and $\{p_5, p_6, p_7, p_8\}$, say $\{c_1, p_5\}$, then we have a green $p_7s_4p_6yp_5c_1p_4c_3$. So, assume that all the edges from R to $\{p_5, p_6, p_7, p_8\}$ are blue. If there is at least one blue edge from $\{p_5, p_6, p_7, p_8\}$ to $\{s_2, s_3\}$,

say $\{p_5, s_2\}$, then we have a blue $p_4xs_1s_2p_5c_1p_6c_3$. In the opposite case we obtain a green $p_5s_2p_6s_3p_7s_4p_8y$.

Case 3. There is a blue edge from x to Q, say $\{x, p_4\}$, all the edges from y to $Q \setminus \{p_4\}$ are green, and $l = 5$ (the edge $\{y, p_4\}$ can be blue or green).

Let the blue P_l in H be $xs_1s_2s_3y$. There is a vertex $z \in H\setminus\{x, y, s_1, s_2, s_3\}$, such that the edges $\{x, z\}$ and $\{y, z\}$ are green, since otherwise $l = 6$. This case is broken into three subcases, as follows:

1. There are at least two blue edges from p_4 to R, say $\{p_4, c_1\}$ and $\{p_4, c_3\}$. When avoiding blue P_8, we obtain a green $P_8 = xzyp_8c_3p_7c_1p_6$.

2. There is exactly one blue edge from p_4 to R, say $\{p_4, c_1\}$. Then, if there is at least one green edge from $\{c_3, c_5\}$ to $Q \setminus \{p_4\}$, say $\{c_3, p_5\}$, then we have green $P_8 = acbp_8c_1p_5c_3p_4$. In the opposite case we must have a blue complete blue bipartite subgraph $K_{\{c_3,c_5\},\{p_5,p_6,p_7,p_8\}}$. If there is at least one blue edge from $\{p_5, p_6, p_7, p_8\}$ to $\{s_1, s_2, s_3\}$, then we have a blue P_8, otherwise we easily find a green P_8.

3. All the edges from p_4 to R are green. Then, if there is at least one green edge from R to $Q\setminus\{p_4\}$, say $\{c_1, p_5\}$, then in order to avoid a green P_8, we must have a blue complete bipartite $K_{\{c_3,c_5\},\{p_6,p_7,p_8\}}$. In the opposite case, we have a complete blue bipartite subgraph $K_{\{c_1,c_3,c_5\},\{p_6,p_7,p_8\}}$. If there is at least one blue edge from $\{p_6, p_7, p_8\}$ to $\{s_1, s_2, s_3\}$, then we have a blue P_8, otherwise we have a green P_8.

Case 4. There is a blue edge from x to Q, say $\{x, p_4\}$, there is a blue edge from y to a different vertex in Q, say $\{y, p_8\}$, and $l = 5$.

Let the blue P_l in H be $xs_1s_2s_3y$. There is a vertex $z \in H\setminus\{x, y, s_1, s_2, s_3\}$, such that the edges $\{x, z\}$ and $\{y, z\}$ are green, since otherwise $l = 6$. All the edges from $\{p_4, p_8\}$ to $R\cup\{z\}$ are green. If there are at least two green edges from a vertex in $\{p_5, p_6, p_7\}$ to R, say $\{p_5, c_1\}$ and $\{p_5, c_3\}$, then we have a green P_8, namely $c_5p_4c_1p_5c_3p_8zy$. In the opposite case, we have a blue P_4, without loss of generality, say $c_1p_5c_3p_6$. To avoid a blue P_8, the edges $\{x, p_6\}$ and $\{s_1, p_6\}$ are green, but then we have a green $P_8 = s_1p_6xzp_4c_1p_8c_5$. $\square$

It is interesting to observe that the case of $R_3(P_8)$ required significantly more complex reasoning than that of $R_3(P_9)$. We also tried to proceed for P_{10} and P_{11} similarly as in the proofs of Theorems 10 and 8, respectively, but while the general method seems to be applicable, the complexity of case analysis appears to be quite harder. We did not complete these proofs. In general, we expect that even paths cases are harder than those for odd paths. Consequently, between the first two open cases of Conjecture 2, namely the questions whether it is true that $R_3(P_{10}) = 18$ and $R_3(P_{11}) = 21$, we expect the former to be more difficult to prove.

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Appendix A

In this appendix we present results obtained by Gyárfás, Rousseau and Schelp [9]. Let a, b, c be positive integers. The authors of [9] answered the question of what is the maximum number of edges $f_0(a, b, c)$ (resp. $f_1(a, b, c)$) in any P_{2l}-free (resp. P_{2l+1}-free), for $l > c$, subgraph of the complete bipartite graph $K_{a,b}$ ($a \leq b$). They also characterized all the corresponding extremal graphs. Adjacency matrix ($a \times b$) of all extremal graphs can be written in partitioned form as

$$G = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where each block in this partitioned matrix is either a matrix of all 1's or a matrix of all 0's. Such a matrix is completely specified by giving the size of M_{11} ($s \times t$), and identifying each M_{ij} as a block of all 1's or all 0's.

Table I (resp. II) describe the constructions of all the extremal graphs achieving $f_0(a, b, c)$ for $c \geq 1$ (resp. $f_1(a, b, c)$ for $c \geq 3$). Table III (resp. IV) present formulas for $f_0(a, b, c)$ for $c \geq 1$ (resp. $f_1(a, b, c)$ for $c \geq 2$). In all four tables it is assumed that $a \leq b$.

Graphs	Range	s	t	M_{11}	M_{12}	M_{21}	M_{22}
G_{01}	$a \leq c$	a	b	1			
G_{02}	$c < a \leq 2c$	c	b	1		0	
G_{03}	$a = 2c$	c	any	1	0	0	1
G_{04}	$a > 2c$	c	$b - c$	1	0	0	1

Table I: Extremal P_{2l}-free subgraphs of $K_{a,b}$, for $l > c \geq 1$ [9].

Graphs	Range	s	t	M_{11}	M_{12}	M_{21}	M_{22}
G_{11}	$a \leq c$ or $a = b = c+1$	a	b	1			
G_{12}	$a = c+1$ and $b = c+2$	$c+1$	$c+1$	1	0		
G_{13}	$a = b$ and $a = c+2$	$c+1$	$c+1$	1	0	0	1
G_{14}	$b > a = c+1$ or $c+1 < a < 2(c+1)$	c	$b-1$	1	1	0	1
G_{15}	$a = 2c+1$ or $a = b = 2(c+1)$	$c+1$	$c+1$	1	0	0	1
G_{16}	$b > a = 2(c+1)$ or $a > 2(c+1)$	c	$b-c$	1	0	0	1

Table II: Extremal P_{2l+1}-free subgraphs of $K_{a,b}$, for $l > c \geq 3$ [9].

In order to make Table II complete for $c = 2$, two more types of extremal graphs need to be defined, and the special case of $c = 1$ also can be easily handled (see [9]). In this paper, for odd path we used only $c \geq 3$.

No.	Range	$f_0(a,b,c)$
1	$a \leq c$	ab
2	$c < a < 2c$	bc
3	$a \geq 2c$	$(a+b-2c)c$

Table III: Formulas for $f_0(a,b,c)$ for even paths, for $c \geq 1$ [9].

No.	Range	$f_1(a,b,c)$
1	$a \leq c$	ab
2	$a = b = c+1$	$(c+1)^2$
3	$b > a = c+1$ or $c+1 < a < 2(c+1)$	$a + (b-1)c$
4	$a = b = 2(c+1)$	$2(c+1)^2$
5	$b > a = 2(c+1)$ or $a > 2(c+1)$	$(a+b-2c)c$

Table IV: Formulas for $f_1(a,b,c)$ for odd paths, for $c \geq 2$ [9].

Appendix B

In this appendix we present the characterization of all extremal graphs achieving the bound $w(n,k)$, as described by Caccetta and Vijayan in [2]. The definition of $w(n,k)$ (3) with related quantities, and Theorem 5 [15, 2] stating the bound can be found in Section 3. These extremal graphs are needed in the proof of our Theorem 8 in Section 4.

Let G be any graph on n vertices and $w(n,t)$ edges, where $t = c(G)$ is the circumference of G, and let $r = (n-1) \mod (t-1)$. Then G is one of the graphs described as follows.

- If $n = t$, then there exists one extremal graph, K_t,
- If $n > t$ and ($r = t/2 - 1$ or $r = t/2$), then there exist two extremal graphs:
  - (1) $K_{t/2} + \overline{K_{n-t/2}}$, and
  - (2) 1-connected graph with $p-1$ cut-vertices between consecutive blocks B_1, B_2,. . . ,B_p, $p = \left\lceil \frac{n-1}{t-1} \right\rceil$, where $p-1$ blocks are equal to the graph K_t and one block to the graph K_{r+1},
- If $n > t$, $r \neq t/2 - 1$ and $r \neq t/2$, then there exists one extremal graph, namely $K_1 + (\left\lfloor \frac{n-1}{t-1} \right\rfloor K_{t-1} \cup K_r)$.