Zarankiewicz Numbers and Bipartite Ramsey Numbers

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Abstract

The Zarankiewicz number $z(b;s)$ is the maximum size of a subgraph of $K_{b,b}$ which does not contain $K_{s,s}$ as a subgraph. The two-color bipartite Ramsey number $b(s,t)$ is the smallest integer b such that any coloring of the edges of $K_{b,b}$ with two colors contains a $K_{s,s}$ in the first color or a $K_{t,t}$ in the second color.

In this work, we design and exploit a computational method for bounding and computing Zarankiewicz numbers. Using it, we obtain several new values and bounds on $z(b;s)$ for $3 \leq s \leq 6$. Our approach and new knowledge about $z(b;s)$ permit us to improve some of the results on bipartite Ramsey numbers obtained by Goddard, Henning and Oellermann in 2000. In particular, we compute the smallest previously unknown bipartite Ramsey number, $b(2,5) = 17$. Moreover, we prove that up to isomorphism there exists a unique 2-coloring which witnesses the lower bound $16 < b(2,5)$. We also find tight bounds on $b(2,2,3)$, $17 \leq b(2,2,3) \leq 18$, which currently is the smallest open case for multicolor bipartite Ramsey numbers.

Keywords: Zarankiewicz number, bipartite Ramsey number
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1 Introduction

Graph notation

If G is a bipartite graph, with the bipartition of its vertices $V(G) = L(G) \cup R(G)$, or simply $V = L \cup R$, we will denote it by writing $G[L, R]$. Furthermore, when we wish to point only to the orders m and n of the *left* and *right* parts of the vertex set $V(G)$, $m = |L|, n = |R|$, we will use notation $G[m, n]$. The parts L and R will be called *left vertices* and *right vertices* of G, respectively. If H is a subgraph of G, and its bipartition is $H[L', R']$, then we will consider only the cases when $L' \subset L$ and $R' \subset R$. For the remainder of this paper, all bipartite graphs have a fixed bipartition.

This allows us to treat any bipartite graph $G[m, n]$ as $m \times n$ 0-1 matrix M_G, whose rows are labeled by L, columns are labeled by R, and where 1's stand for the corresponding edges between L and R. The (bipartite) reflection of G is obtained by swapping the left and right vertices of G, or equivalently by transposing the corresponding 0-1 matrix. The bipartite complement $\overline{G}$ of a (bipartite) graph $G[L, R]$, has the same bipartition as G, but its matrix representation is the binary complement of M_G.

Zarankiewicz numbers

The *Zarankiewicz number* $z(m, n; s, t)$ is defined to be the maximum number of edges in any subgraph $G[m, n]$ of the complete bipartite graph $K_{m,n}$, such that $G[m, n]$ does not contain $K_{s,t}$. For the diagonal cases, we will use $z(m, n; s)$ and $z(n; s)$ to denote $z(m, n; s, s)$ and $z(n, n; s, s)$, respectively.

In 1951, Kazimierz Zarankiewicz [26] asked what is the minimum number of 1's in a 0-1 matrix of order $n \times n$, which guarantees that it has a 2×2 minor of 1's. In the notation introduced above, it asks for the value of $z(n, n; 2, 2) + 1$.

General Zarankiewicz numbers $z(m, n; s, t)$ and related extremal graphs have been studied by numerous authors, including Kővári, Sós, and Turán [17], Reiman [21], Irving [16], and Goddard, Henning, and Oellermann [12]. A nice compact summary of what is known was presented by Bollobás [3] in 1995. Recently, Füredi and Simonovits [11] published an extensive survey of relationships between $z(m, n; s, t)$ and much studied Turán numbers $\mathrm{ex}(k, K_{s,t})$.

The results and methods used to compute or estimate $z(n; 2)$ are similar to those in the widely studied case of $\mathrm{ex}(n, C_4)$, where one seeks the maximum number of edges in any C_4-free n-vertex graph. Previous papers established the exact values of $z(n; s)$ for all $n \leq 21$ [7], and some recent as of yet unpublished work by Afzaly and McKay pushed it further to all $n \leq 31$ [1], see also Table 3 in the Appendix. Early papers by Irving [16] and Roman [22] presented some bounding methods and results for concrete cases with $s > 2$. For more data for $3 \leq s \leq 6$ see our Appendix. For detailed discussion of general bounds and asymptotics, especially for $s = 2$ and $s = 3$, see the work by Füredi and Simonovits [11].

Bipartite Ramsey numbers

The *bipartite Ramsey number* $b(s_1, \ldots, s_k)$ is the least positive integer b such that any coloring of the edges of the complete bipartite graph $K_{b,b}$ with k colors contains K_{s_i,s_i} in the i-th color for some i, $1 \leq i \leq k$.

If $s_i = s$ for all i, then we will denote this number by $b_k(s)$. The study of bipartite Ramsey numbers was initiated by Beineke and Schwenk in 1976, and continued by others, in particular Exoo [8], Hattingh and Henning [15], Goddard, Henning, and Oellermann [12], and Lazebnik and Mubayi [18].

The connection between Zarankiewicz numbers and bipartite Ramsey numbers is quite straightforward: the edges in color i in any coloring of $K_{n,n}$ witnessing $n < b(s_1, \ldots, s_k)$ give a lower bound witness for $e \leq z(n, n; s_i)$, where the i-th color has e edges. Thus, upper bounds on $z(n; s)$ can be useful in obtaining upper bounds on bipartite Ramsey numbers. This relationship was originally exploited by Irving [16], developed further by several authors, including Goddard, Henning, and Oellermann [12], and it will be used in this paper. The role of Zarankiewicz numbers and witness graphs in the study of bipartite Ramsey numbers is very similar to that of Turán numbers $\mathrm{ex}(n, G)$ and G-free graphs in the study of classical Ramsey numbers, where we color the edges of K_n while avoiding G in some color.

For multicolor bipartite cases ($k > 2$), we know most when avoiding C_4, i.e. for $s = 2$. The following exact values have been established: $b_2(2) = 5$ [2], $b_3(2) = 11$ [8], and $b_4(2) = 19$ [24, 7]. In the smallest open case for 5 colors it is known that $26 \leq b_5(2) \leq 28$ [7], where the lower bound was obtained by a 5-coloring of $GF(5^2) \times GF(5^2)$, and the upper bound is implied by a general upper bound on $z(k^2 + k - 2)$ for $k = 5$. It was also conjectured that $b_5(2) = 28$ [7].

Finally, we wish to point to the work by Fenner, Gasarch, Glover and Purewal [9], who wrote a very extensive survey of the area of grid colorings, which are equivalent to edge colorings of complete bipartite graphs. Their focus is on the cases avoiding C_4 for both Zarankiewicz and Ramsey problems.

Notes on asymptotics

Asymptotics of Zarankiewicz numbers is quite well understood (relative to Ramsey numbers). The classical bound by Kövári, Sós, and Turán [17], generalized by several authors (cf. [3, 11]) is

$$z(m, n; s, t) < (s - 1)^{1/t}(n - t + 1)m^{1-1/t} + (t - 1)m,$$

which for constant $s = t$ becomes $z(n; s) = O(n^{2-1/t})$. Füredi [10] improved the general bound to the best known so far

$$z(m, n; s, t) < (s - t + 1)^{1/t}(n - t + 1)mn^{1-1/t} + (t - 1)n^{2-2/t} + (t - 2)m,$$

for $m \geq t$ and $n \geq s \geq t \geq 2$. These upper bounds are asymptotically optimal, as discussed in a book chapter by Bollobás [3], and a more recent monograph by Füredi and Simonovits [11].

In 2001, Caro and Rousseau [4], using upper bounds on Zarankiewicz numbers $z(n, n; s, s)$, proved that for any fixed $m \geq 2$ there exist constants A_m and B_m such that for sufficiently large n we have

$$A_m \Big(\frac{n}{\log n}\Big)^{(m+1)/2} < b(m, n) < B_m \Big(\frac{n}{\log n}\Big)^{m}.$$

The asymptotics of other off-diagonal cases, including avoidance of $K_{s,t}$ and other bipartite graphs, was studied by Lin and Li [19], and others. For the diagonal case, the best known asymptotic upper bound $b(n, n) < \big(1+o(1)\big)2^{n+1} \log_2 n$ was obtained by Conlon [6].

Overview of this paper

In the remainder of this paper we consider only the case of avoiding balanced complete $K_{s,s}$, i.e. the case of $s = t$. Thus, for brevity, in the following the Zarankiewicz numbers will be written as $z(m, n; s)$ or $z(n; s)$.

The main contribution of this paper is the method for computing and bounding $z(m, n; s)$ for small $s > 2$, and the results obtained by using it. The background to the method and the method itself are presented in Section 2. The main results of this paper and the computations leading to them are presented in Section 3. The results are as follows: We obtain several new values and bounds on $z(n; s)$ for $3 \leq s \leq 6$. We compute the smallest previously unknown bipartite Ramsey number, $b(2, 5) = 17$, and we prove that up to isomorphism there exists a unique 2-coloring which witnesses the lower bound $16 < b(2, 5)$. Finally, we find tight bounds on $b(2, 2, 3)$, $17 \leq b(2, 2, 3) \leq 18$, which currently is the smallest open case for multicolor bipartite Ramsey numbers.

2 Two lemmas and their applications

The focus of this section is on Lemmas 2 and 3, and refining their applications. In the context of bipartite Ramsey numbers and Zarankiewicz numbers, these lemmas may be found in various forms in Section 12 of [13], in [16], [12], and [7]. For use throughout the paper, we introduce the following notation.

Definition 2.1. *Let $G[m, n]$ be some bipartite graph. For positive integers e, s, t, we say that*

1. *G is a (m, n, e^+)-graph if $e(G) \geq e$,*
2. *G is a $(m, n, e^+)_s$-graph if G is a (m, n, e^+)-graph and $K_{s,s} \not\subseteq G$, and*
3. *G is a $(m, n, e^+)_{s,t}$-graph if G is a $(m, n, e^+)_s$-graph and $K_{t,t} \not\subseteq \overline{G}$.*

In any of the above notations, we may replace "e^+" with "e" whenever the condition $e(G) \geq e$ is strengthened to $e(G) = e$ and drop "$, e^+$" whenever no restriction is placed on $e(G)$.

For example, "$z(m,n;s) \geq z$" is equivalent to "there exists some $(m,n,z)_s$-graph" and "$b(s,t) \geq m+1$" is equivalent to "there exists some $(m,m)_{s,t}$-graph". In general, we will use the placeholder $\mathcal{P}$ to denote any of the empty word, "s", and "s,t".

Proposition 1. *For fixed positive integers p,k,t, among the k-part sums $a_1 + \cdots + a_k = p$ with $a_i \geq 0$, the sum*

$$\sum_{i=1}^{k} \binom{a_i}{t} \tag{1}$$

is minimized when $|a_i - a_j| \leq 1$ for all $1 \leq i < j \leq k$.

Proof. Suppose $a_1 + \cdots + a_k = p$ is not balanced in the above sense, and assume without loss of generality that $a_1 \geq \cdots \geq a_k$. It follows that $a_1 - a_k > 1$. Let $b_1 := a_1 - 1$, $b_2 = a_2, \cdots, b_{k-1} = a_{k-1}, b_k := a_k + 1$, be a new split of p into k parts. Note that $\sum_{i=1}^{k} \binom{a_i}{t} - \binom{b_i}{t} = \binom{a_1-1}{t-1} - \binom{a_k}{t-1} \geq 0$, thus a more balanced k-part sum does not increase (1). Consequently, (1) is minimized for some sum as stated. $\square$

Lemma 2 (Star-Counting Lemma)**.** *Let G be a $(m,n,e^+)_s$-graph with $e(G) = md_L + r_L = nd_R + r_R$, where $0 \leq r_L < m$ and $0 \leq r_R < n$. If (a_i) is the left degree sequence of G (and thus $a_1 + \cdots + a_m = e(G)$), then*

$$(m - r_L) \cdot \binom{d_L}{s} + r_L \cdot \binom{d_L + 1}{s} \leq \sum_{i=1}^{m} \binom{a_i}{s} \leq (s-1) \cdot \binom{n}{s}. \tag{2}$$

Proof. Fix G as above. The first inequality in (2) follows from Proposition 1 since the leftmost expression corresponds to a balanced m-part composition of e. Note that the middle expression of (2) counts the number of stars $K_{1,s} \subseteq G$ whose center is on the left. If this sum exceeds $(s-1) \cdot \binom{n}{s}$, then by the pigeonhole principle, there is some $B \subseteq R$, with $|B| = s$ which is the set of leaves of at least s of the left stars as above. However, their union must contain a $K_{s,s}$ subgraph of G, so the second inequality holds as well. $\square$

An analogous statement holds for the right-hand side of G.

Lemma 3 (Density Lemma)**.** *Let G be $(m,n,e^+)_{\mathcal{P}}$-graph and $f = e - \lfloor e/m \rfloor$. Then G contains an induced $(m-1,n,f^+)_{\mathcal{P}}$-subgraph.*

Proof. Fix G as above and let $\overline{d} = e/m$. Since $\overline{d}$ is the average left degree, we may find and remove some left vertex of degree at most $\overline{d}$, leaving us with a $(m-1,n,f^+)_{\mathcal{P}}$-graph. $\square$

An analogous statement holds for the right-hand side of G. Lemma 2 provides a static upper bound $z(m,n;s) \leq e$ based on the parameters m,n,s,e alone. Likewise, given an upper bound on $z(m-1,n;s)$, Lemma 3 gives an upper bound on $z(m,n;s)$.

Example 2.1. *We will show that there is no $(4,4,10)_2$-graph. The most balanced possible composition of* 10 *into* 4 *parts is* $2+2+3+3 = 10$. *For* $m = n = 4$, $s = 2$ *and* $d_L = r_L = 2$ *we have the left- and right-hand side of* (2) *equal to* 8 *and* 6*, respectively. Thus,* $z(4;2) \leq 9$.

Example 2.2. *We will show that there is no $(4,5,12+)_2$-graph. If we apply Lemma* 3 *to any $(5,4,12)_2$-graph, then we obtain a $(4,4,10^+)$-graph. As argued in the previous example such graphs do not exist, hence* $z(4,5;2) \leq 11$.

We combine Lemmas 2 and 3 as follows.

Lemma 4. *Let m, n, s, w be positive integers. If* $z(m-1,n;s) < z \leq w - \lfloor w/m \rfloor$, *then* $z(m,n;s) < w$. *Also, if* $a_1 + \cdots + a_m = w$ *satisfies* $|a_i - a_j| \leq 1$ *for all* $1 \leq i < j \leq m$ *and* $\binom{a_1}{s} + \cdots + \binom{a_m}{s} > (s-1)\binom{n}{s}$, *then* $z(m,n;s) < w$.

Suppose we know that $W = (w_{ij})_{i,j=1}^{m,n}$ are upper bounds on the Zarankiewicz numbers $z(i,j;s)$. Then, we may be able to improve some of them using Lemma 4 by traversing the indices (i,j), $1 \leq i \leq m$, $1 \leq j \leq n$ in some order from $(1,1)$ to (m,n) so that Lemma 4 can be applied at each step. Call this algorithm z_bound.

Backwards paths extensions

The bounds found just by z_bound alone can be often improved by exhaustive methods. If this is successful for any parameters, further application of the z_bound algorithm can lead to improvements for higher parameters. The same technique will be used to bound bipartite Ramsey numbers. In both cases, we will attempt to construct all $(m,n,e^+)_{\mathcal{P}}$-graphs (a possibly empty set). To do this, we begin with all $(a,b,f^+)_{\mathcal{P}}$-graphs, where a,b,f are chosen carefully. For convenience, write

$$(m,n,e^+)_{\mathcal{P}} \sqsupset (a,b,f^+)_{\mathcal{P}}$$

if it is known that any $(m,n,e^+)_{\mathcal{P}}$-graph contains some induced $(a,b,f^+)_{\mathcal{P}}$-subgraph. When $(a,b) = (m-1,n)$ or $(a,b) = (m,n-1)$, the "$\sqsupset$" will be called a *step*. A *backwards path* is a sequence of steps, such as

$$(m_k, n_k, e_k^+) \sqsupset (m_{k-1}, n_{k-1}, e_{k-1}^+) \sqsupset \cdots \sqsupset (m_0, n_0, e_0^+).$$

We aim at constructing the set of all $(m_k,n_k,e_k^+)_{\mathcal{P}}$-graphs, up to bipartite graph isomorphism, using the following extend algorithm. First, generate all of the $(m_0,n_0,e_0^+)_{\mathcal{P}}$-graphs up to bipartite graph isomorphism by some other method. Now suppose we have all $(m_i,n_i,e_i^+)_{\mathcal{P}}$-graphs. For each such graph G, generate $(m_{i+1},n_{i+1},e_{i+1}^+)$-graphs by adding a new vertex v to the appropriate side of degree $d(v) \geq e_{i+1} - e(G)$ in all possible ways. In addition, if $\mathcal{P} =$"s" or "s,t" and the bound $z \geq z(m_{i+1},n_{i+1};s)$ is known, we may also impose a condition that $d(v) \leq z - e(G)$. Remove all generated graphs which are not of type $(m_{i+1},n_{i+1},e_{i+1}^+)_{\mathcal{P}}$. Reduce the remaining set up to bipartite graph isomorphism, which can be readily accomplished by using McKay's nauty package [20]. Repeat this for all G until all (m_k,n_k,e_k^+)-graphs are generated.

Question 2.1. *Given $a, b, m, n, e, \mathcal{P}$, how can we find a suitable backwards path $(m, n, e^+)_{\mathcal{P}} \sqsupset \cdots \sqsupset (a, b, f^+)_{\mathcal{P}}$ in such a way that f is as large as possible?*

In principle, one could use techniques from dynamic programming to obtain all such optimal paths, yet this is not practical in our case. While Lemmas 2 and 3 are easy to apply at all times, the question whether it is feasible to run extend algorithm depends on other unpredictable factors.

Example 2.3. *In order to aid in the computation of the bipartite Ramsey number $b(2, 5)$ and characterization of its lower bound witnesses, we found a backwards path $(16, 16, 189^+)_{5,2} \sqsupset \cdots \sqsupset (7, 7, f^+)_{5,2}$, which is displayed in Figure* 1*, where the rows and columns correspond to i and j ranging from 7 to 16. For reconstruction, the starting $(7, 7, 42+)_{5,2}$-graphs have the number of edges close to the maximum equal to $z(7; 5) = 44$ (see Table* 6*).*

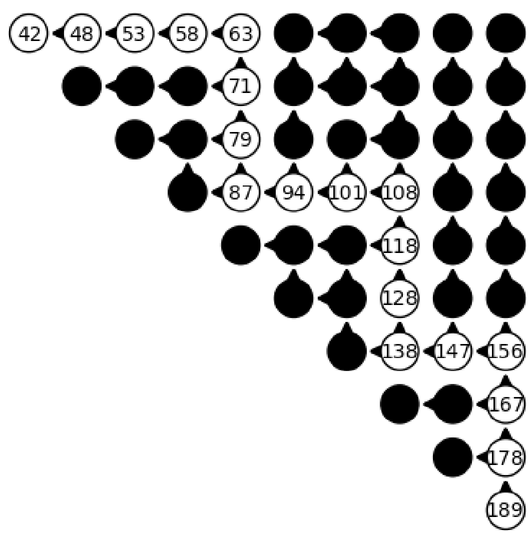

Figure 1: Backwards path witnessing $(16, 16, 189^+)_{5,2} \sqsupset (7, 7, 42^+)_{5,2}$.

The path highlighted in Figure 1 illustrates the sensitive nature of this process. Parity plays a crucial role and it is not obvious to the authors how in general to find a backwards path from (m, n, e^+) to the destination (a, b, f^+) maximizing f and feasible to follow with computations. The pointers from entries indicate which immediately smaller parameters where considered when performing computations leading to the displayed path. If instead we step backwards along the main diagonal, we end at $(7, 7, 39^+)_{5,2}$. Stepping back in two straight paths (straight to entry $(7, 16)$, then straight to entry $(7, 7)$) coincidentally gives the same end value 39. Note that a bare density comparison gives only $(16, 16, 189^+)_{5,2} \sqsupset (7, 7, \lceil 49 \cdot 189/256 \rceil^+)_{5,2} = (7, 7, 37^+)_{5,2}$. Up to isomorphism, there are 7500 $(7,7,37^+)_{5,2}$-graphs, 1619 $(7,7,39^+)_{5,2}$-graphs, but only 33 $(7,7,42^+)_{5,2}$-graphs.

3 Bipartite Ramsey Numbers and Sidon Sets

We motivate this section with two results from [5]. When trying to establish the lower bound $16 < b(2,5)$, one may consider searching for witness graphs which satisfy certain structural properties. The nauty package command

```
genbg 16 16 64:64 -d4:4 -D4:4 -Z1
```

lists, up to isomorphism, all 4-regular $(16,16,64)_2$-graphs where the neighborhood of any two vertices intersect in at most one neighbor (so that the graphs are $K_{2,2}$-free). This runs in a few minutes on an ordinary laptop computer, and produces 19 graphs. After removing the graphs whose bipartite complement contains a $K_{5,5}$, a single $(16,16,64)_{2,5}$-graph remains and its bipartite adjacency matrix is shown in Table 1.

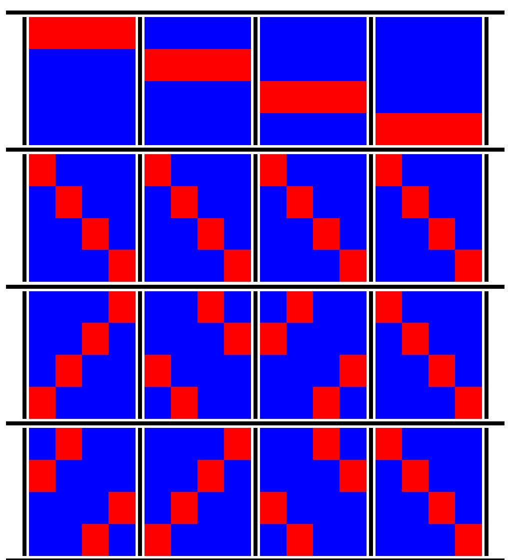

Table 1: The bipartite adjacency matrix of a $16 < b(2,5)$ witness.

There is clear structure in this $(16,16,64)_{2,5}$-graph. The matrix of Table 1 is a 4×4 arrangement of 4×4 blocks, 12 of them being permutation matrices of 4 elements. This graph has $2304 = 2^8 3^2$ automorphisms.

In [5], a cyclic witness on 15 vertices to the 3-color bipartite Ramsey number $b(2,2,3)$ was found, but no witness of any kind could be found on 16 vertices.

Question 3.1. *Are the bounds* $16 < b(2,5)$ *and* $15 < b(2,2,3)$ *tight?*

In Theorem 6 of the next section, we will be able to conclude that $b(2,5) = 17$ and that the lower bound witness found in Table 1 is indeed the unique witness. For the second part of Question 3.1 we will be able to improve the lower bound by 1 using bipartite Cayley graphs as described in the remaining part of this section.

Our definition of bipartite Cayley graphs generalizes the classical cyclic constructions. Given Γ a group and $S \subseteq \Gamma$ a set of *edge generators*, the *bipartite Cayley graph generated by* S is $X(\Gamma, S)$, where $V(X(\Gamma, S)) = \Gamma \times \{1, 2\}$ and for each $g \in \Gamma$ and $s \in S$, there is an edge between $(g, 1)$ and $(g \cdot s, 2)$. We impose no restriction on the symmetry of S and accept the identity 1_Γ as a valid edge generator. We can easily describe what causes $X(\Gamma, S)$ to avoid $K_{2,2}$ using the concept of Sidon sets.

Definition 3.1. *Given a group* Γ*, a subset* $S \subseteq \Gamma$ *is* Sidon *if there are no solutions in* S *to*

$$s_1 s_2^{-1} s_3 s_4^{-1} = 1_\Gamma,$$

unless $s_i = s_{i+1}$ *for some* $i = 0, 1, 2, 3$*, with indices taken modulo* 4.

Sidon sets were originally defined over the integers, while the above is a well-known generalization to arbitrary groups. For a more detailed discussion, see [14] and [25].

Proposition 5. $X(\Gamma, S)$ *is* $K_{2,2}$*-free if and only if* S *is Sidon.*

Proof. Suppose $S \subseteq \Gamma$ has a solution $s_1, s_2, s_3, s_4 \in S$ to $s_1 s_2^{-1} s_3 s_4^{-1} = 1_\Gamma$. Then with $a := 1_\Gamma$, $b := s_1$, $c := s_1 s_2^{-1}$, and $d := s_1 s_2^{-1} s_3 = s_4$, note that $a \neq c$ and $b \neq d$. It follows that $K_{2,2} \subseteq X(\Gamma, S)$. Conversely, suppose there is some $K_{2,2} \subseteq X(\Gamma, S)$ with left vertices a, c and right vertices b, d. Then setting $s_1 := a^{-1}b$, $s_2 := c^{-1}b$, $s_3 := c^{-1}d$ and $s_4 := a^{-1}d$, we see that $s_1, s_2, s_3, s_4 \in S$ satisfies $s_1 s_2^{-1} s_3 s_4^{-1} = 1_\Gamma$. Assuming $a \neq c$ and $b \neq d$ (because of $s_i \neq s_{i+1}$), this is genuinely a $K_{2,2}$. $\square$

The 3-color construction on 15 vertices in [5] witnessing $15 < b(2,2,3)$ can be described in terms of Sidon sets as follows: Let Γ be the additive group modulo 15, $\mathbb{Z}_{15}$, and consider three bipartite Cayley graphs $X(\mathbb{Z}_{15}, S_i)$, $0 \leq i \leq 2$, where $S_0 = \{0, 1, 3, 7\}, S_1 = \{2, 4, 12, 13\}$, and $S_2 = \{5, 6, 8, 9, 10, 11, 14\}$. One can check that S_0 and S_1 are Sidon, S_2 yields a $K_{3,3}$-free graph, and that the edges of $X(\mathbb{Z}_{15}, S_i)$'s partition the edges of $K_{15,15}$.

We searched for witnesses to $16 < b(2,5)$ and $16 < b(2,2,3)$ using the same principle. Our approach was to search among the bipartite Cayley graphs whose edge generators are Sidon sets and whose groups are of order 16. This task is not difficult, since `sage` [23] has a page[1] which lists groups of small order alongside commands to generate them. A few lines of code in `sage` allowed us

[1] http://doc.sagemath.org/html/en/constructions/groups.html#construction-instructions-for-every-group-of-order-less-than-32

to automate the process of generating the Sidon sets S in a group. Checking whether the bipartite complement $X(\Gamma, \Gamma \setminus S)$ contains $K_{5,5}$ can also be done easily in sage. We may assume that $1_\Gamma \in S$ without loss of generality. Among the 14 groups of order 16, the only group which produced a desired construction was Dic_4, the *dicyclic* group of order 16 (a generalization of the quaternion group). Up to isomorphism, only a single bipartite Cayley graph of the form $X(\Gamma, S)$ witnessing $16 < b(2,5)$ was found. Interestingly, this graph is isomorphic to the one in Table 1. Based on the same Sidon set, a 3-colored adjacency matrix of $K_{16,16}$ corresponding to the bound $16 < b(2,2,3)$ is presented in Table 2.

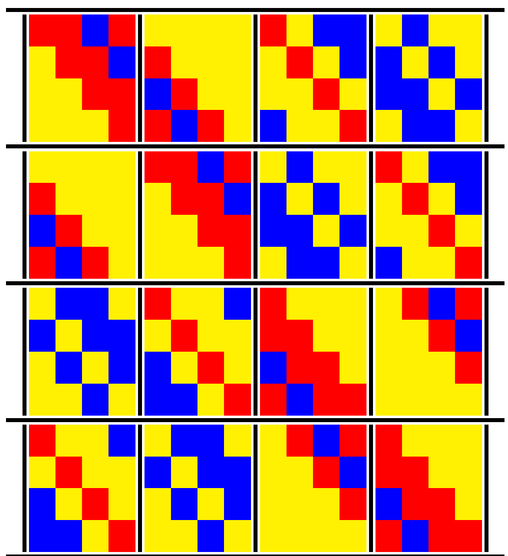

Table 2: A 3-color bipartite adjacency matrix witnessing $16 < b(2,2,3)$.

4 Main Results

Theorem 6. *We have that*

$$b(2,5) = 17,$$

and the unique witness to $16 < b(2,5)$ *is the* $(16,16)_{2,5}$*-graph given by Table* 1*. Moreover, the only way to realize this witness as a bipartite Cayley graph is with the dicyclic group* Dic_4*.*

Proof. The lower bound is implied by the constructions discussed in previous section. The conclusion $b(2,5) = 17$ follows by inspection of Tables 3 and 6 in the Appendix: $z(17;2) = 74$ and $z(17;5) \leq 213$, adding to 287, which is not sufficient to cover all 289 edges of $K_{17,17}$. The uniqueness of the constructed graph on 16 vertices follows from a similar but more detailed argument. Since

$z(16;2) = 67$, it suffices to consider all of the $(16, 16, 189^+)_{5,2}$-graphs. All of such graphs were generated using the algorithms described in Section 2 along the computational backwards path displayed in Figure 1. The final computation (after performing many auxiliary computations and consistency verifications) terminates in under a half hour on an ordinary laptop computer. It returns a single graph $G[16, 16]$ with 192 edges, isomorphic to the graph given by Table 1. We wish to note that a seemingly much simpler approach of considering all potential $(16, 16, 64^+)_{2,5}$-graphs resulted to be computationally infeasible using our methods. $\square$

Theorem 7. *It holds that*

$$17 \leq b(2, 2, 3) \leq 18.$$

Proof. The lower bound witness is found in Table 2. The upper bound is implied by using the bounds in Tables 3 and 4 in the Appendix: $z(18;2) = 81$, $z(18;3) \leq 156$, and $2 \cdot 81 + 156 = 318 < 324$. $\square$

Conjecture 8. *We conjecture that* $b(2, 2, 3) = 17$.

At present, Tables 3 and 4 are quite close to providing a proof: $z(17;2) = 74$ and $z(17;3) \leq 141$, so we have that $2 \cdot 74 + 141 = 289$ is just barely too large. It would suffice to prove that $z(17;3) \leq 140$, but our computational attempts to obtain this bound have proven to be too time-consuming. The interested reader may note other weak-looking bounds in Table 4, such as for $z(k, 17;3)$ for $13 \leq k \leq 17$.

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Appendix: Small Zarankiewicz Numbers

The problem of computing $z(m, n;2)$ is well-studied (cf. [13], [7], [1]). Below in Table 3, we only list the values of $z(n;2)$ until the first open case at $n = 32$. More details on $z(m, n;2)$ and related cases can be found in a recent work by Afzaly and McKay [1].

n	$z(n;2)$	n	$z(n;2)$	n	$z(n;2)$
1	1	12	45	23	115
2	3	13	52	24	122
3	6	14	56	25	130
4	9	15	61	26	138
5	12	16	67	27	147
6	16	17	74	28	156
7	21	18	81	29	165
8	24	19	88	30	175
9	29	20	96	31	186
10	34	21	105	32	189/190
11	39	22	108	33	

Table 3: Zarankiewicz numbers $z(n;2)$ from [7] and [1].

The following are tables of upper bounds on some small Zarankiewicz numbers. A boldfaced entry is an exact value. A superscript * indicates that there exists a unique $(m, n, z(m, n; s))_s$-graph. A superscript † indicates that there is also a unique $(m, n, z(m, n; s)-1)_s$-graph. An italicized entry indicates that the bound or value was determined with exhaustive computations. Otherwise, an undecorated number indicates that the bound was obtained by using Lemmas 2, 3 and 4, and without exhaustive enumeration of $(m, n, e^+)_s$-graphs.

	6	7	8	9	10	11	12	13	14	15	16	17	18
6	**26***	**29**	**32**	**36***	**39***	**42**	**45***	**48***	**50**	**53**	**56**	**58**	61
7		**33***	**37***	**40**	**44***	**47**	**50**	**53**	**56**	**60***	**63***	**66**	69
8			**42***	**45**	**50***	**53**	**57***	**60**	**64***	**67**	**70**	**74***	78
9				**49**	**54**	**59***	**64***	**67***	**70**	**73**	**77**	**81**	85
10					**60**†	**64***	**68**	**73***	**77**	**81***	**85***	**90***	94
11						**69***	**74**	**80**	**84**	**88**	**92**	**96**	101
12							**80**	**86***	**91***	**96**	**99**	**103***	109
13								**92***	**98***	**104***	**107**	*110*	116
14									**105***	**112***	**115***	118	124
15										**120**†	**123***	126	132
16											**128***	*133*	140
17												141	148
18													156

Table 4: Bounds on Zarankiewicz numbers $z(m, n; 3)$.

	6	7	8	9	10	11	12	13	14	15	16	17	18
6	**31***	**36***	**39**	**43**	**47**	**51**	**55**	**59**	**63**	**67**	**71***	**75***	**78**
7		**42**†	**45**	**49**	**54**	**58**	**63**	**68***	**72**	**77**	**82***	**87***	**90**
8			**51***	**55**	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95***	**99**
9				**61**	**67**	**72**	**78***	**84***	**88**	**94**	**99**	**104**	*109*
10					**74***	**79**	**86***	**93***	**97**	**103**	**109**	**115**	**120**
11						**86**	**93***	**100***	**105**	**111**	*117*	124	131
12							101	109	114	121	127	134	141
13								118	123	131	137	145	152
14									132	141	147	156	163
15										151	157	166	174
16											167	177	185
17												188	196
18													207

Table 5: Bounds on Zarankiewicz numbers $z(m, n; 4)$.

	6	7	8	9	10	11	12	13	14	15	16	17	18
6	**33***	**38***	**43***	**48***	**52**	**57**	**62**	**67***	**72***	**76**	**81**	**86**	91
7		**44***	**50***	**56***	**60**	**66**	**72**	**78***	**84**†	**88**	**92***	**96**	101
8			**57***	**64***	**68**	**74**	**80**	**86***	**92***	**97**	**103**	**109**	115
9				**72**†	**76**	**82**	**88**	**95**	**101**	**108**	**114**	**121**	128
10					**84***	**90**	**97**	**104**	**110**	**117**	**124**	**131**	138
11						**98**	**106**	**113**	**120**	**127**	**135***	**142**	150
12							**114**	**122**	**130**	**138**	*146*	**154***	163
13								**132***	**140**	**149***	*156*	*165*	174
14									**150***	**160***	*168*	*177*	187
15										**171***	*180*	*189*	200
16											**192**†	*201*	212
17												*213*	225
18													238

Table 6: Bounds on Zarankiewicz numbers $z(m, n; 5)$.

	6	7	8	9	10	11	12	13	14	15	16	17	18
6	**35***	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
7		**46***	**52***	**58***	**64***	**70***	**75**	**81**	**87**	**93**	**99***	**105***	**110**
8			**59***	**66***	**73***	**80***	**85**	**92**	**99**	**106**	**113***	**120***	**125**
9				**74***	**82***	**90***	**95**	**102**	**109**	**116**	**123**	**130**	**137**
10					**95***	**100***	**105**	**112**	**120**	**127**	**135**	**142**	**150**
11						**110***	**115**	**122**	**130**	**138**	**147**	**155**	**163**
12							**125***	**132**	**141**	**150***	**158**	**167**	**176**
13								**142**	**152***	**161**	**170**	**180**	**189**
14									**162**	**172**	*182*	**192***	*202*
15										**184***	195	205	216
16											208	218	230
17												231	244
18													258

Table 7: Bounds on Zarankiewicz numbers $z(m, n; 6)$.

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