Computers in Ramsey Theory

testing, constructions and nonexistence

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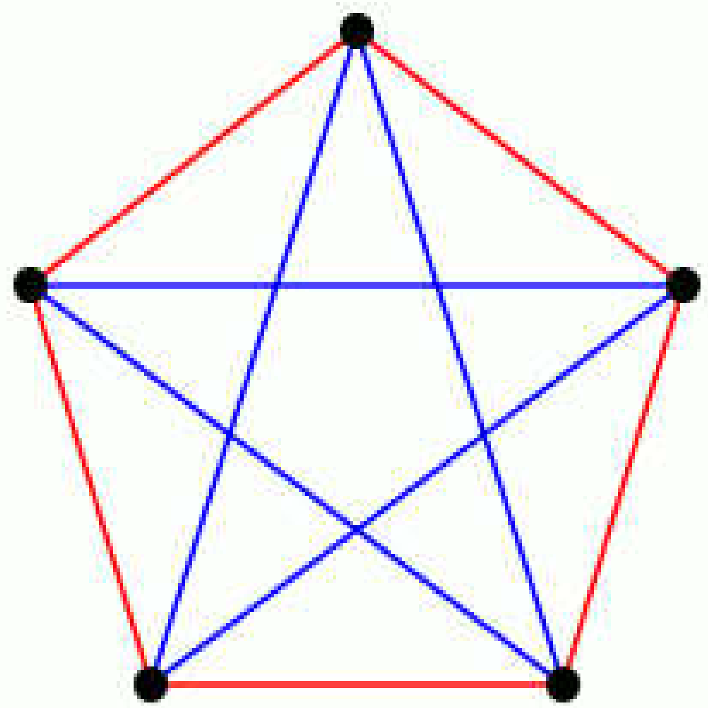
Ramsey Numbers

- $R(G, H) = n$ iff
  minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color.

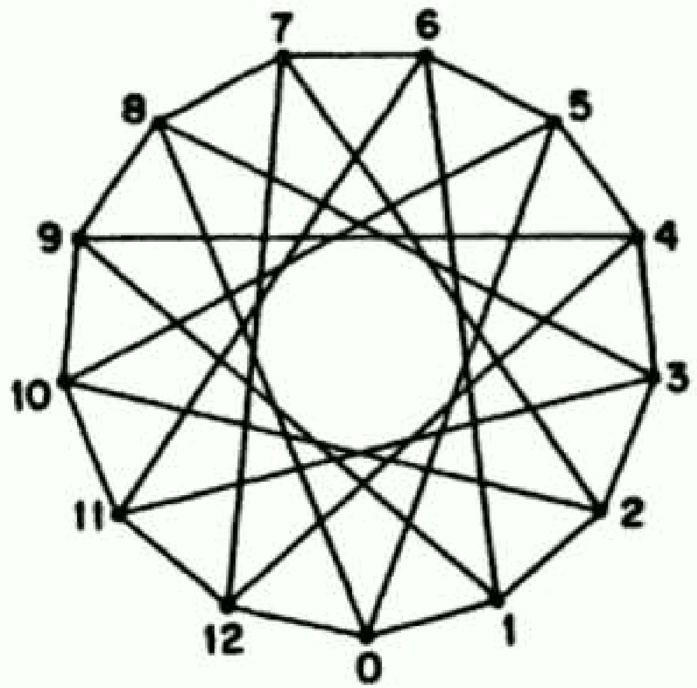
- $2 - colorings \cong graphs$, $R(m, n) = R(K_m, K_n)$
- Generalizes to k colors, $R(G_1, \cdots, G_k)$

- Theorem (Ramsey 1930): Ramsey numbers exist

Unavoidable classics



$R(3,3) = 6$



$R(3,5) = 14$ [GRS'90]

Asymptotics

diagonal cases

- **Bounds** (Erdős 1947, Spencer 1975; Conlon 2010)

$$\frac{\sqrt{2}}{e} 2^{n/2} n < R(n,n) < R(n+1,n+1) \leq \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, \$100)

  $\lim_{n\to\infty} R(n,n)^{1/n}$ exists.

  If it exists, it is between $\sqrt{2}$ and 4 (\$250 for value).

Asymptotics

Ramsey numbers avoiding K_3

- Kim 1995, lower bound
  Ajtai-Komlós-Szemerédi 1980, upper bound

$$R(3,n) = \Theta\left(\frac{n^2}{\log n}\right)$$

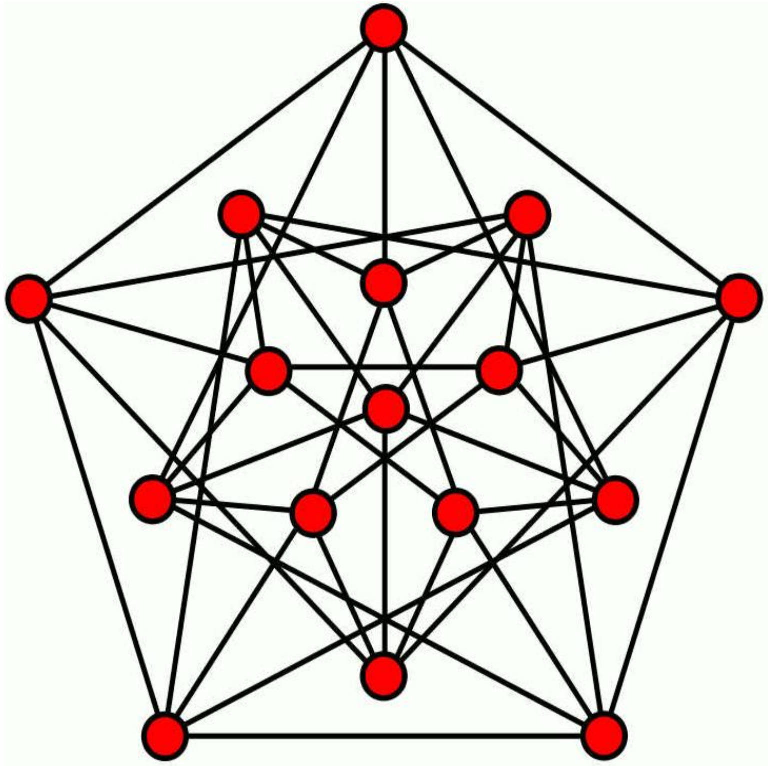
- Bohman/Keevash 2009/2013, triangle-free process
- Fiz Pontiveros-Griffiths-Morris, lower bound, 2013
  Shearer, upper bound, 1983

$$\left(\frac{1}{4} + o(1)\right) n^2/\log n \leq R(3,n) \leq (1 + o(1))n^2/\log n$$

Clebsch $(3, 6; 16)$-graph on $GF(2^4)$

$(x, y) \in E$ iff $x - y = \alpha^3$

[Wikipedia]



Alfred Clebsch (1833-1872)

#vertices / #graphs

no exhaustive searches beyond 13 vertices

3	4
4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
11	1018997864
12	165091172592
13	50502031367952 $\approx 5 * 10^{13}$
	too many to process
14	29054155657235488 $\approx 3 * 10^{16}$
15	31426485969804308768
16	64001015704527557894928
17	245935864153532932683719776
18	$\approx 2 * 10^{30}$

Test - Hunt - Exhaust

Ramsey numbers

- **Testing:** do it right.
  Graph G is a witness of $R(m, n) > k$ iff
  $|V(G)| = k$, $cl(G) < m$ and $\alpha(G) < n$.

  Lab in a 200-level course.

- **Hunting:** constructions and heuristics.
  Given m and n, find a witness G for k larger than others.
  Challenge projects in advanced courses.

  Master: Geoffrey Exoo 1986–

- **Exhausting:** generation, pruning, isomorphism.
  Prove that for given m, n and k, there does not exist any witness as above. Hard without `nauty/traces`.

  Master: Brendan McKay 1991–

Values and bounds on $R(m, n)$

two colors, avoiding K_m, K_n

l / k	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 42	47 50	53 59	60 68	67 77	74 87
4		18	25	36 41	49 61	59 84	73 115	92 149	102 191	128 238	138 291	147 349	155 417
5			43 48	58 87	80 143	101 216	133 316	149 442	183 633	203 848	233 1138	267 1461	269 1878
6				102 165	115 298	134 495	183 780	204 1171	256 1804	294 2566	347 3703	5033	401 6911
7					205 540	217 1031	252 1713	292 2826	405 4553	417 6954	511 10578	15263	22112
8						282 1870	329 3583	343 6090	10630	16944	817 27485	41525	865 63609
9							565 6588	581 12677	22325	38832	64864		
10								798 23556	45881	81123			1265

[SPR, EIJC survey *Small Ramsey Numbers*, revision #15, 2017, with updates]

Small $R(m, n)$ bounds, references

two colors, avoiding K_m, K_n

l / k	4	5	6	7	8	9	10	11	12	13	14	15
3	GG	GG	Kéry	Ka2 GrY	GR McZ	Ka2 GR	Ex5 GoR1	Ex20 GoR1	Kol1 Les	Kol1 GoR1	Kol2 GoR1	Kol2 GoR1
4	GG	Ka1 MR4	Ex19 MR5	Ex3 Mac	ExT Mac	Ex16 Mac	HaKr1 Mac	ExT Spe4	SuLL Spe4	ExT Spe4	ExT Spe4	ExT Spe4
5		Ex4 AnM	Ex9 HZ1	CaET HZ1	HaKr1 Spe4	Kuz Mac	ExT Mac	Kuz HW+	Kuz HW+	Kuz HW+	Kuz HW+	ExT HW+
6			Ka2 Mac	ExT HZ1	ExT Mac	Kuz Mac	Kuz Mac	Kuz HW+	Kuz HW+	Kuz HW+	HW+	2.3.h HW+
7				She2 Mac	XSR2 HZ1	Kuz HZ2	Kuz Mac	XXER HW+	XSR2 HW+	XuXR HW+	HW+	HW+
8					BurR Mac	Kuz Ea1	Kuz HZ2	HW+	HW+	XXER HW+	HW+	2.3.h HW+
9						She2 ShZ1	XSR2 Ea1	HW+	HW+	HW+		
10							She2 Shi2	HW+	HW+			2.3.h

[ElJC survey *Small Ramsey Numbers*, revision #15, 2017]

Small $R(m, n)$, references

$R(5, 5) \leq 48$, Angeltveit-McKay 2017.

k \ l	4	5	6	7	8	9	10	11	12	13	14	15
3	GG	GG	Kéry	Ka2 GrY	GR McZ	Ka2 GR	Ex5 GoR1	Ex20 GoR1	Kol1 Les	Kol1 GoR1	Kol2 GoR1	Kol2 GoR1
4	GG	Ka1 MR4	Ex19 ~~MR5~~	Ex3 ~~Mac~~	ExT ~~Mac~~	Ex16 ~~Mac~~	HaKr1 ~~Mac~~	ExT ~~Spe4~~	SuLL ~~Spe4~~	ExT ~~Spe4~~	ExT ~~Spe4~~	ExT ~~Spe4~~
5		Ex4 AnM	Ex9 ~~HZ1~~	CaET ~~HZ1~~	HaKr1 ~~Spe4~~	Kuz ~~Mac~~	ExT ~~Mac~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	ExT ~~HW+~~
6			Ka2 ~~Mac~~	ExT ~~HZ1~~	ExT ~~Mac~~	Kuz ~~Mac~~	Kuz ~~Mac~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	~~HW+~~	2.3.h ~~HW+~~
7				She2 ~~Mac~~	XSR2 ~~HZ1~~	Kuz ~~HZ2~~	Kuz ~~Mac~~	XXER ~~HW+~~	XSR2 ~~HW+~~	XuXR ~~HW+~~	~~HW+~~	~~HW+~~
8					BurR ~~Mac~~	Kuz ~~Exl~~	Kuz ~~HZ2~~	~~HW+~~	~~HW+~~	XXER ~~HW+~~	~~HW+~~	2.3.h ~~HW+~~
9						She2 ~~ShZ1~~	XSR2 ~~Exl~~	~~HW+~~	~~HW+~~	~~HW+~~		
10							She2 ~~Shi2~~	~~HW+~~	~~HW+~~			2.3.h

Spring 2017 avalanche of improved upper bounds
after LP attack for higher m and n by Angeltveit-McKay.

Small $R(K_m, C_n)$

	C_3	C_4	C_5	C_6	C_7	C_8	C_9	...	C_n for $n \geq m$
K_3	6 GG-Bush	7 ChaS	9 ...	11	13	15	17	... ...	$2n-1$ ChaS
K_4	9 GG	10 ChH2	13 He4/JR4	16 JR2	19 YHZ1	22 ...	25	... ...	$3n-2$ YHZ1
K_5	14 GG	14 Clan	17 He2/JR4	21 JR2	25 YHZ2	29 BolJY+	33 ...	... ...	$4n-3$ BolJY+
K_6	18 Kéry	18 Ex2-RoJa1	21 JR5	26 Schi1	31 ...	36	41	... ...	$5n-4$ Schi1
K_7	23 Ka2-GrY	22 RaT-JR1	25 Schi2	31 CheCZN	37 CheCZN	43 JaBa/Ch+	49 Ch+	... ...	$6n-5$ Ch+
K_8	28 GR-McZ	26 RaT	29-33 JaAl2	36 ChenCX	43 ChenCZ1	50 JaAl1/ZZ3	57 BatJA	... ...	$7n-6$ conj.
K_9	36 Ka2-GR	30 RaT-LaLR					65 conj.	... ...	$8n-7$ conj.
K_{10}	40-42 Ex5-GoR1	36 LaLR						...	$9n-8$ conj.
K_{11}	47-50 Ex20-GoR1	39-44 LaLR						...	$10n-9$ conj.

Erdős-Faudree-Rousseau-Schelp 1976 conjecture:
$R(K_m, C_n) = (m-1)(n-1)+1$ for all $n \geq m \geq 3$, except $m = n = 3$.

Lower bound witness: complement of $(m-1)K_{n-1}$.

First two columns: $R(3, m) = \Theta(m^2/\log m)$,
$c_1(m^{3/2}/\log m) \leq R(K_m, C_4) \leq c_2(m/\log m)^2$.

Known bounds on $R(3, K_s)$ and $R(3, K_s - e)$

$J_s = K_s - e$, $\Delta_s = R(3, K_s) - R(3, K_{s-1})$

s	$R(3, J_s)$	$R(3, K_s)$	Δ_s	s	$R(3, J_s)$	$R(3, K_s)$	Δ_s
3	5	6	3	10	37	40–42	4–6
4	7	9	3	11	42–45	47–50	5–10
5	11	14	5	12	47–53	53–59	3–12
6	17	18	4	13	55–62	60–68	3–13
7	21	23	5	14	60–71	67–77	3–14
8	25	28	5	15	69–80	74–87	3–15
9	31	36	8	16	74–91	82–97	3–16

$R(3, J_s)$ and $R(3, K_s)$, for $s \leq 16$

(Goedgebeur-R 2014, SRN 2017)

Conjecture

and $1/2$ of Erdős-Sós problem

Observe that
$R(3, s+k) - R(3, s-1) = \sum_{i=0}^{k} \Delta_{s+i}$.

We know that
$\Delta_s \geq 3$, $\Delta_s + \Delta_{s+1} \geq 7$, $\Delta_s + \Delta_{s+1} + \Delta_{s+2} \geq 11$.

Conjecture
There exists $d \geq 2$ such that $\Delta_s - \Delta_{s+1} \leq d$ for all $s \geq 2$.

Theorem
If Conjecture is true, then $\lim_{s \to \infty} \Delta_s / s = 0$.

52 Years of $R(5,5)$

year	reference	lower	upper	
1965	Abbott	38		quadratic residues in $\mathcal{Z}_{37}$
1965	Kalbfleisch		59	pointer to a future paper
1967	Giraud		58	LP
1968	Walker		57	LP
1971	Walker		55	LP
1973	Irving	42		sum-free sets
1989	Exoo	43		simulated annealing
1992	McKay-R		53	$(4,4)$-graph enumeration, LP
1994	McKay-R		52	more details, LP
1995	McKay-R		50	implication of $R(4,5) = 25$
1997	McKay-R		49	long computations
2017	Angeltveit-McKay		48	massive LP for $(\geq 4, \geq 5)$-graphs

History of bounds on $R(5,5)$

$43 \leq R(5,5) \leq 48$

Conjecture. McKay-R 1997
$R(5,5) = 43$, and the number of $(5,5;42)$-graphs is 656.

- $42 < R(5,5)$:
  - Exoo's construction of the first $(5,5;42)$-graph, 1989.
  - Any new $(5,5;42)$-graph would have to be in distance at least 6 from all 656 known graphs, McKay-Lieby 2014.
- $R(5,5) \leq 48$, Angeltveit-McKay 2017:
  - Enumeration of all 352366 $(4,5;24)$-graphs.
  - Overlaying pairs of $(4,5;24)$-graphs, and completing to any potential $(5,5;48)$-graph, using intervals of cones.
  - Similar technique for the new bound $R(4,6) \leq 40$.

$R(4,4;3) = 13$

2-colorings of 3-uniform hypergraphs avoiding monochromatic tetrahedrons

- The only non-trivial classical Ramsey number known for hypergraphs, McKay-R 1991.
- Enumeration of all valid 434714 two-colorings of triangles on 12 points. $K_{13}^{(3)} - t$ cannot be thus colored, McKay 2016.
- For size Ramsey numbers, the above gives

  $$\widehat{R}(4,4;3) \leq 285 = \binom{13}{3} - 1,$$

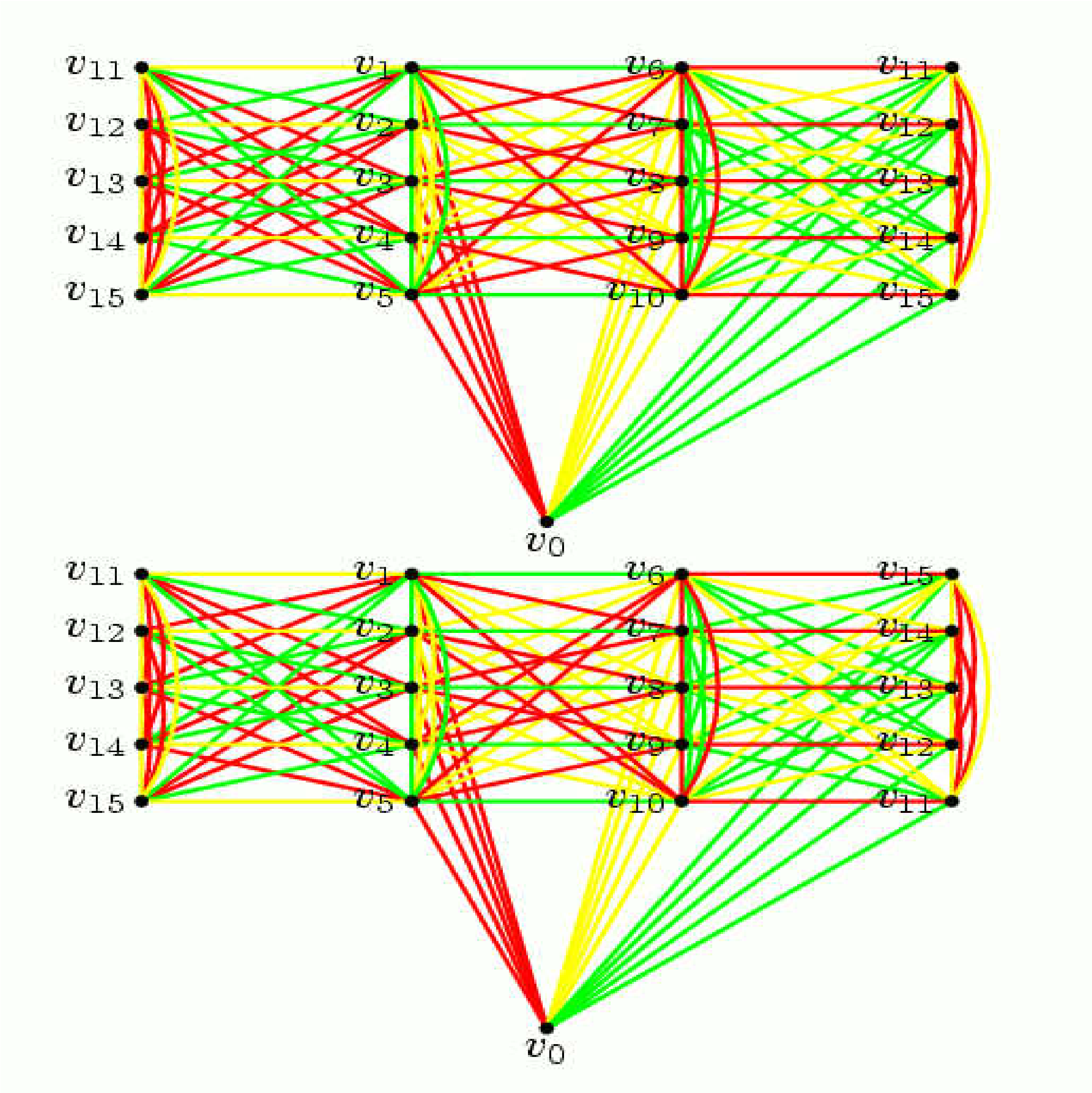
  which answers in negative a general question posed by Dudek, La Fleur, Mubayi and Rödl, 2015.

$R_r(3) = R(3,3,\cdots,3)$

- Much work on Schur numbers $s(r)$
  via sum-free partitions and cyclic colorings
  $s(r) > 89^{r/4 - c\log r} > 3.07^r$ [except small r]
  Abbott+ 1965+
- $s(r) + 2 \leq R_r(3)$
- $R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3$
  Chung 1973
- The limit $L = \lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$ exists
  Chung-Grinstead 1983
  $(2s(r)+1)^{\frac{1}{r}} = c_r \approx_{(r=6)} 3.199 < L$

$R(3,3,3) = 17$

two Kalbfleisch $(3,3,3;16)$-colorings, each color is a Clebsch graph



[Wikipedia]

Four colors - $R_4(3)$

$51 \leq R(3,3,3,3) \leq 62$

year	reference	lower	upper
1955	Greenwood, Gleason	42	66
1967	false rumors	[66]	
1971	Golomb, Baumert	46	
1973	Whitehead	50	65
1973	Chung, Porter	51	
1974	Folkman		65
1995	Sánchez-Flores		64
1995	Kramer (no computer)		62
2004	Fettes-Kramer-R (computer)		62

History of bounds on $R_4(3)$ [from FKR 2004]

Four colors - $R_4(3)$

color degree sequences for $(3, 3, 3, 3; \geq 60)$-colorings

n	orders of $N_\eta(v)$	
65	[16, 16, 16, 16]	Whitehead, Folkman 1973-4
64	[16, 16, 16, 15]	Sánchez-Flores 1995
63	[16, 16, 16, 14]	
	[16, 16, 15, 15]	
62	[16, 16, 16, 13]	Kramer 1995+
	[16, 16, 15, 14]	–
	[16, 15, 15, 15]	Fettes-Kramer-R 2004
61	[16, 16, 16, 12]	
	[16, 16, 15, 13]	
	[16, 16, 14, 14]	
	[16, 15, 15, 14]	
	[15, 15, 15, 15]	
60	[16, 16, 16, 11]	guess: doable in 2017
	[16, 16, 15, 12]	
	[16, 16, 14, 13]	
	[16, 15, 15, 13]	
	[16, 15, 14, 14]	
	[15, 15, 15, 14]	

- Why don't heuristics come close to $51 \leq R_4(3)$?
- Improve on $R_4(3) \leq 62$

Diagonal Multicolorings for Cycles

Bounds on $R_k(C_m)$ in 2017 SRN

k \ m	3	4	5	6	7	8
3	17	11	17	12	25	16
4	51 62	18	33 137	18 20	49	20
5	162 307	27 29	65	26	97	28
6	538 1838	34 43	129		193	

Table XIII. Known values and bounds for $R_k(C_m)$ for small k, m;

Columns:

- 3 - just triangles, the most studied
- 4 - relatively well understood, thanks geometry!
- 5 - bounds on $R_4(C_5)$ have a big gap

What to do next?

computationally

- A nice, open, intriguing, feasible to solve case (Exoo 1991, Piwakowski 1997)

$$28 \leq R_3(K_4 - e) \leq 30$$

- improve on $20 \leq R(K_4, C_4, C_4) \leq 22$
- improve on $27 \leq R_5(C_4) \leq 29$
- improve on $33 \leq R_4(C_5) \leq 137$

Folkman Graphs and Numbers

For graphs F, G, H and positive integers s, t

- $F \to (s, t)^e$ iff in every 2-coloring of the edges of F there is a monochromatic K_s in color 1 or K_t in color 2
- $F \to (G, H)^e$ iff in every 2-coloring of the edges of F there is a copy of G in color 1 or a copy of H in color 2
- variants: coloring vertices, more colors

Edge Folkman graphs
$\mathcal{F}_e(s, t; k) = \{F \mid F \to (s, t)^e,\ K_k \not\subseteq F\}$

Edge Folkman numbers
$F_e(s, t; k)$ = the smallest order of graphs in $\mathcal{F}_e(s, t; k)$

Theorem (Folkman 1970)
If $k > \max(s, t)$, then $F_e(s, t; k)$ and $F_v(s, t; k)$ exist.

Test - Hunt - Exhaust

Folkman numbers

Hints.

- Inverted role of lower/upper bounds wrt Ramsey
- F_e tends to be much harder than F_v

Folkman is harder then Ramsey.

- Testing: $F \to (G, H)$ is Π_2^p-complete,
  only some special cases run reasonably well.
- Hunting: Use smart constructions.
  Very limited heuristics.
- Exhausting: Do proofs.
  Currently, computationally almost hopeless.

Bounds from Chromatic Numbers

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$, $M = R(a_1, \cdots, a_r)$.

Theorem (Nenov 2001, Lin 1972, others)

If $G \rightarrow (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \rightarrow (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

Special Case of Folkman Numbers

is just about graph chromatic number $\chi(G)$

Note: $G \to (2 \cdots_r 2)^v \iff \chi(G) \geq r+1$

For all $r \geq 1$, $F_v(2^r; 3)$ exists and it is equal to
the smallest order of $(r+1)$-chromatic triangle-free graph.

$F_v(2^{r+1}; 3) \leq 2F_v(2^r; 3) + 1$, Mycielski construction, 1955

small cases

$F_v(2^2; 3) = 5$, C_5, Mycielskian, 1955

$F_v(2^3; 3) = 11$, the Grötzsch graph, Mycielskian, 1955

$F_v(2^4; 3) = 22$, Jensen and Royle, 1995

$32 \leq F_v(2^5; 3) \leq 40$, Goedgebeur, 2017

50 Years of $F_e(3,3;4)$

What is the smallest order n of a K_4-free graph
which is not a union of two triangle-free graphs?

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal
1970	exist	Folkman
1972	10 –	Lin
1975	– 10^{10}?	Erdős offers \$100 for proof
1986	– 8×10^{11}	Frankl-Rödl, almost won
1988	– 3×10^{9}	Spencer, won \$100
1999	16 –	Piwakowski-R-Urbański, implicit
2007	19 –	R-Xu
2008	– 9697	Lu, eigenvalues
2008	– 941	Dudek-Rödl, maxcut-SDP
2012	– 100?	Graham offers \$100 for proof
2014	– 786	Lange-R-Xu, maxcut-SDP
2016	20 – 785	Bikov-Nenov / Kaufmann-Wickus-R

Most Wanted Folkman Number: $F_e(3,3;4)$

and how to earn \$100 from RL Graham

The best known bounds:

$$20 \leq F_e(3,3;4) \leq 785.$$

- Upper bound 785 from a modified residue graph via SDP.
- Ronald Graham Challenge for \$100 (2012):
  Determine whether $F_e(3,3;4) \leq 100$.

  Conjecture (Exoo, around 2004):
  - $G_{127} \to (3,3)^e$, moreover
  - removing 33 vertices from G_{127} gives graph G_{94},
    which still looks good for arrowing, if so, worth \$100.
- Lower bound: very hard, crawls up slowly 10 (Lin 1972),
  16 (PUR 1999), 19 (RX 2007), 20 (Bikov-Nenov 2016).

Graph G_{127}

Hill-Irving 1982, a cool K_4-free graph studied as a Ramsey graph

$$G_{127} = (\mathcal{Z}_{127}, E)$$
$$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$$

Exoo conjectured that $G_{127} \rightarrow (3, 3)^e$.

- resists direct backtracking
- resists eigenvalues method
- resists semi-definite programming methods
- resists state-of-the-art 3-SAT solvers
- amazingly rich structure,
  hence perhaps will not resist a proof by hand ...

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Other Computational Approaches

each with some success

- Huele, 2005–17: SAT-solvers, VdW numbers, Pythagorean triples, *Science of Brute Force*, CACM August 2017.
- Codish, Frank, Itzhakov, Miller (2016): finishing $R(3,3,4) = 30$, symmetry breaking, BEE (Ben-Gurion Equi-propagation Encoder) to CNF, CSP.
- Lidický-Pfender (2017), using Razborov's flag algebras (2007) for 2- and 3-color upper bounds.
- Surprising new lower bounds by heuristics: Kolodyazny, Kuznetsov, Exoo, Tatarevic (2014–2017).
- Ramsey quantum computations, D-Wave? (2020–).

Papers to look at

- SPR, revision #15 of the survey paper
  Small Ramsey Numbers at the *EIJC*, March 2017.

- Xiaodong Xu and SPR,
  Some Open Questions for Ramsey and Folkman Numbers,
  in *Graph Theory, Favorite Conjectures and Open Problems*,
  Problem Books in Mathematics
  Springer 2016, 43–62.

- Rujie Zhu, Xiaodong Xu, SPR,
  A small step forwards on the Erdős-Sós problem concerning the Ramsey numbers $R(3, k)$,
  DAM 214 (2016), 216–221.

Thanks for listening!