On the Nonexistence of Some Generalized Folkman Numbers*

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Abstract

For an undirected simple graph G, we write $G \to (H_1, H_2)^v$ if and only if for every red-blue coloring of its vertices there exists a red H_1 or a blue H_2. The generalized vertex Folkman number $F_v(H_1, H_2; H)$ is defined as the smallest integer n for which there exists an H-free graph G of order n such that $G \to (H_1, H_2)^v$. The generalized edge Folkman numbers $F_e(H_1, H_2; H)$ are defined similarly, when colorings of the edges are considered.

We show that $F_e(K_{k+1}, K_{k+1}; K_{k+2}-e)$ and $F_v(K_k, K_k; K_{k+1}-e)$ are well defined for $k \geq 3$. We prove the nonexistence of $F_e(K_3, K_3; H)$ for some H, in particular for $H = B_3$, where B_k is the book graph of k triangular pages, and for $H = K_1 + P_4$. We pose three problems on generalized Folkman numbers, including the existence question of edge Folkman numbers $F_e(K_3, K_3; B_4)$, $F_e(K_3, K_3; K_1 + C_4)$ and $F_e(K_3, K_3; \overline{P_2 \cup P_3})$. Our results lead to some general inequalities involving two-color and multicolor Folkman numbers.

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1 Introduction

Let G be a finite undirected graph that contains no loops or multiple edges. Denote by $V(G)$ the set of its vertices and $E(G)$ the set of its edges. For vertex-disjoint graphs G and H, the join graph $G+H$ has the set of vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H) \cup \{\{(u, v\} \mid u \in V(G), v \in V(H)\}$. For a set of vertices $S \subset V(G)$, $G[S]$ is the graph induced by S in G, and $G-u$ is the graph obtained from G by removing vertex $u \in V(G)$ together with all the edges adjacent to u.

The complete graph of order n is denoted by K_n, and a cycle of length n by C_n. The book graph B_k is defined as $K_1 + K_{1,k}$, and the complete graph K_n with one missing edge will be denoted by J_n. The clique number of G will be denoted by $cl(G)$, and the chromatic number of G by $\chi(G)$. An (s,t)-graph is a graph that does not contain K_s neither any independent sets of t vertices. The set $\{1, \cdots, n\}$ will be denoted by $[n]$.

For graph G, we write $G \rightarrow (H_1, H_2)^v$ if and only if for every red-blue coloring χ of the vertices $V(G)$ there exists a red subgraph H_1 or a blue subgraph H_2 in χ. The generalized vertex Folkman number $F_v(H_1, H_2; H)$ is defined as the smallest integer n for which there exists an H-free graph G of order n such that $G \rightarrow (H_1, H_2)^v$. The set of all H-free graphs satisfying the latter vertex arrowing will be denoted by $\mathcal{F}_v(H_1, H_2; H)$.

The generalized edge Folkman numbers $F_e(H_1, H_2; H)$ are defined similarly, when colorings of the edges are considered. We write $G \rightarrow (H_1, H_2)^e$ if and only if for every red-blue coloring χ of the edges $E(G)$ there exists a red subgraph H_1 or a blue subgraph H_2 in χ. The generalized edge Folkman number $F_e(H_1, H_2; H)$ is defined as the smallest integer n for which there exists an H-free graph G of order n such that $G \rightarrow (H_1, H_2)^e$. The set of all H-free graphs satisfying the latter edge arrowing will be denoted by $\mathcal{F}_e(H_1, H_2; H)$.

The cases when H_1, H_2 and H are complete graphs have been studied by many authors, for two and more colors, in particular in [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 15, 17]. Often, if the graphs H_i and H are complete, we will simply write the order of the graph, say, as in $F_e(s, t; k)$ instead of $F_e(K_s, K_t; K_k)$. In this paper we focus on two colors, but we will also make some comments related to more colors, such as in commonly studied multicolor vertex Folkman numbers $F_v(a_1, a_2, \cdots, a_r; s)$ and edge Folkman numbers $F_e(a_1, a_2, \cdots a_r; s)$, where a_i's are the orders of the arrowed complete graphs while coloring K_s-free graphs. We note that the classical Ramsey number $R(a_1, \cdots, a_r)$ can be defined as the smallest integer n such that $K_n \rightarrow (a_1, \cdots, a_r)^e$. In the diagonal case $a_1 = \cdots = a_r = a$ we may use a more compact notation $F_v^r(a; s) = F_v(a_1, \cdots, a_r; s)$ and $\mathcal{F}_v^r(a; s) = \mathcal{F}_v(a_1, \cdots, a_r; s)$, similarly $F_e^r(a; s) = F_e(a_1, \cdots, a_r; s)$ and $\mathcal{F}_e^r(a; s) = \mathcal{F}_e(a_1, \cdots, a_r; s)$, as well as for arrowing general graphs, such as in $F_e^r(G; H)$.

In 1970, Folkman [5] proved that for any integer $s > \max\{a_1, \cdots, a_r\}$, both sets $\mathcal{F}_v(a_1, \cdots, a_r; s)$ and $\mathcal{F}_e(a_1, a_2; s)$ are nonempty, and thus the corresponding Folkman numbers are well defined. In 1976, Nešetřil and Rödl [12] generalized this result to the multicolor edge cases, namely they proved that the sets $\mathcal{F}_e(a_1, \cdots, a_r; s)$ are also nonempty, for arbitrary $r \geq 2$ and $s > \max\{a_1, \cdots, a_r\}$. An interesting upper bound on $F_v^r(a; s)$ was obtained by Dudek and Rödl [2] in 2010, as in the following theorem.

Theorem 1. [2] *For any positive integer r there exists a constant $C = C(r)$ such that for every $s \geq 2$ it holds that $F_v^r(s; s+1) \leq Cs^2 \log^4 s$.*

The above determines that both vertex and edge Folkman numbers exist when the arrowed and avoided graphs are complete, for $s > \max\{a_1, \cdots, a_r\}$. By simple monotonicity, this easily extends to some cases (say, when the arrowed graphs H_i have at most a_i vertices), but apparently it poses interesting existence questions in other cases. Only some special parameters are discussed in the literature, such as the bound $F_e(K_4 - e, K_4 - e; K_4) \leq 30193$ obtained by Lu [8] in 2008. In this paper we focus on some general situations, in particular when the avoided graph H is not the complete graph K_s, but H_1 and H_2 are complete, and often $H_1 = H_2 = K_3$.

The nonexistence of a Folkman number with some parameters is equivalent to the emptiness of the corresponding set of Folkman graphs. For example, the Folkman number $F_e(K_3, K_3; K_1 + P_4)$ does not exist if and only if $\mathcal{F}_e(K_3, K_3; K_1 + P_4) = \emptyset$, which in fact we prove to be true in Theorem 8, Section 4.

The summary of contents of the remainder of this paper is as follows: Vertex and edge arrowing by $(K_s - e)$-free graphs and related existence questions are discussed in Section 2, similarly for graphs involving book graphs in Section 3. Other cases involving wheels and paths are analyzed in Section 4. Finally, in Section 5 some results for more than two colors are presented.

2 Ramsey arrowing by $(K_n - e)$-free graphs

Recall from the introduction that $J_k = K_k - e$. One can easily see that $F_v(2, 2; 3) = F_v(K_2, K_2; K_3) = 5$, which can be equivalently stated as that the smallest number of vertices in any triangle-free graph G with $\chi(G) > 2$ is equal to 5. However, it is also easy to observe that $F_v(K_2, K_2; J_3)$ does not exist, since every J_3-free graph is bipartite. Similarly, we see that $F_e(K_3, K_3; J_4)$ does not exist, since in any J_4-free graph no two triangles can share an edge, and thus the edges of every triangle can be independently red-blue colored. These observations lead to our first theorem.

Theorem 2. *For $k \geq 3$, if the edge Folkman number $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ exists, then the vertex Folkman number $F_v(K_k, K_k; J_{k+1})$ exists too.*

Proof. Suppose that $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ exists, it is equal to n, and let G be any graph of order n in $\mathcal{F}_e(K_{k+1}, K_{k+1}; J_{k+2})$. For any vertex $u \in V(G)$ we must have $G - u \not\rightarrow (K_{k+1}, K_{k+1})^e$. Fix any vertex $u \in V(G)$, and let H be the graph induced in G by the neighbors of u, $H = G[N(u)]$. Clearly, H is a J_{k+1}-free graph.

For contradiction, assume that $F_v(K_k, K_k; J_{k+1})$ does not exist. This implies that $H \not\rightarrow (k, k)^v$, and hence there exists a partition of $N(u)$ into $U_1 \cup U_2$ such that both $G[U_1]$ and $G[U_2]$ are K_k-free. Next, observe that any red-blue edge coloring witnessing $G - u \not\rightarrow (K_{k+1}, K_{k+1})^e$ can be extended to whole $E(G)$, without creating any monochromatic K_{k+1}, by coloring the edges $\{\{u, v\} \in E(G) \mid v \in U_1\}$ red and coloring the edges $\{\{u, v\} \in E(G) \mid v \in U_2\}$ blue. This contradicts that $G \in \mathcal{F}_e(K_{k+1}, K_{k+1}; J_{k+2})$, and completes the proof. □

Graph H is called a Ramsey graph for K_n if $H \rightarrow (K_n, K_n)^e$. In 1981, Nešetřil and Rödl [13] proved the following theorem.

Theorem 3. [13] *Let $n \geq 3$ be a fixed positive integer. Then there exists a Ramsey graph H for K_n such that any two subgraphs K, K' of H isomorphic to K_n intersect in at most two points.*

Corollary 4. *For every integer $k \geq 3$,*
(a) *the edge Folkman number $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ exists, and*
(b) *the vertex Folkman number $F_v(K_k, K_k; J_{k+1})$ exists.*

Proof. Graph H in Theorem 3 does not contain J_{n+1} for $n \geq 4$, thus if $n = k+1$ then the set $\mathcal{F}_e(K_{k+1}, K_{k+1}; J_{k+2})$ is nonempty, and hence part (a) of the corollary follows. Theorem 2 and part (a) imply part (b). □

We can easily see that for integers s and t, if $k > s \geq t \geq 2$, then $F_v(K_s, K_t; J_{k+1})$ exists, and by monotonicity $F_v(K_s, K_t; J_{k+1}) \leq F_v(s, t; k)$. The upper bound for $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ which can be obtained using the proof of Theorem 3 is large, and likely it is much larger than the exact value. Similarly, the implied upper bound for $F_v(K_k, K_k; J_{k+1})$ is likely much larger than the exact value. It would be interesting to obtain better upper bounds for these numbers directly without using Theorem 3, for example by a method similar to one used in the proof of Theorem 1 in [2]. We note that a straightforward reasoning similar to a method used in [6] leads to an inequality $F_v(K_{s_1 s_2}, K_{t_1 t_2}; J_{k_1 k_2 + 1}) \leq F_v(s_1, t_1; k_1 + 1) F_v(K_{s_2}, K_{t_2}; J_{k_2+1})$, for $2 \leq s_1 \leq t_1 \leq k_1$ and $3 \leq s_2 \leq t_2 \leq k_2$. This makes us anticipate that $F_v(K_k, K_k; J_{k+1})$ grows slowly with k, and possibly can be bounded by $cF_v(k, k; k+1)$ for some constant $c > 0$.

The best known concrete lower and upper bounds on various Ramsey numbers of the form $R(J_s, K_t)$ are collected in [14]; for example, we know that $30 \leq R(J_5, K_5) \leq 33$. In that case, any 29-vertex witness graph to Ramsey lower bound seems to be a good candidate for the vertex Folkman number case of arrowing $(3, 4)^v$. This would give an interesting bound $F_v(K_3, K_4; J_5) \leq 29$ (unfortunately, we were not successful in finding any such graph so far). Still we think that, in general, further exploration of witnesses to lower bounds for

Ramsey numbers as graphs showing upper bounds for (vertex or edge) Folkman numbers is worth an effort.

3 Arrowing triangles by B_k-free graphs

Recall that the book graph B_k was defined as $B_k = K_1 + K_{1,k}$, hence it has $k+2$ vertices and consists of k triangles sharing one common edge. In particular, $B_1 = K_3$, $B_2 = J_4$ and $B_3 = K_5 \setminus K_3$. Thus, the first book-specific case (different from K_k and J_k) is that for the book graph B_3 considered in the next theorem.

Theorem 5. *There exists a B_3-free and K_4-free graph G of order 19 such that $G \to (3,3)^v$. Thus we have $F_v(K_3, K_3; B_3) \leq 19$.*

Note. For the upper bound in the second part of the theorem it is not required that the graph G is K_4-free. In any case, we consider the bound in Theorem 5 quite strong. Finding the actual value of $F_v(K_3, K_3; B_3)$ can be difficult, and it is open whether the best construction must contain K_4.

Proof. We will construct the required graph G on the vertex set $V(G) = \bigcup_{i=0}^{3} V_i \cup \{u\}$, where $G[V_0] = K_3$, and the subgraphs induced by V_i are isomorphic to C_5, for $i \in \{1, 2, 3\}$. Let $V_0 = \{v_1, v_2, v_3\}$. The other edges of G are all possible edges between u and $\bigcup_{i=1}^{3} V_i$, and all possible edges between v_i and the vertices in V_i, for each $i \in \{1, 2, 3\}$. Thus G has 19 vertices and 48 edges.

It is easy to see that the graph G is both K_4-free and B_3-free. With little more effort, one can show that every red-blue coloring of $V(G)$ contains a monochromatic triangle. Without loss of generality we can assume that u is red and at least one of the vertices in V_0, say v_1, is blue. However, in order to avoid a red triangle on u and two vertices in V_1, $G[V_1]$ must contain a blue K_2. But the latter together with v_1 would form a blue triangle. Hence we have that $G \to (3,3)^v$. Finally, the same graph G is a witness of the upper bound. $\square$

Since $B_2 = J_4$, and using the observation from the beginning of Section 2, we see that $F_e(K_3, K_3; B_2)$ does not exist. Now we will consider the existence of $F_e(K_3, K_3; B_k)$ for $k \geq 3$, starting with the case of B_3.

Theorem 6. *The edge Folkman number $F_e(K_3, K_3; B_3)$ does not exist.*

Proof. Suppose that $F_e(K_3, K_3; B_3)$ exists, it is equal to n, and let G be any graph of order n in $\mathcal{F}_e(K_3, K_3; B_3)$. For any vertex $u \in V(G)$ we must have $G - u \not\to (K_3, K_3)^e$. Fix any vertex $u \in V(G)$, and let H be the graph induced in G by the neighbors of u, $H = G[N(u)]$. Since G is B_3-free, H does not contain $K_{1,3}$, or equivalently has maximum degree at most 2. Therefore any connected component of H is bipartite or it is an odd cycle.

We will show that any red-blue coloring χ of the edges of $G - u$, such that χ is without monochromatic triangles, can be extended to G without creating any monochromatic triangles. This will contradict the definition of G and thus it will complete the proof.

For the edges $\{u, v\}$, where v is in a bipartite component of H, we assign the color red or blue according to which part of the bipartition v belongs to. For vertices v on odd cycles in H, we proceed as follows. Let U be the vertex set of some odd cycle in H. We can partition U into $U_1 \cup U_2$ so that $H[U_1]$ has exactly one edge, say e, and U_2 is an independent set in H. If $\chi(e)$ is red (blue), then we color the edges in $\{\{u, v\} \mid v \in U_1\}$ blue (red), and the edges in $\{\{u, v\} \mid v \in U_2\}$ red (blue). □

We were not able to answer the question whether $F_e(K_3, K_3; B_4)$ exists, and hence we leave it as an open problem for the readers. Note that for every $k \geq 5$, the edge Folkman number $F_e(K_3, K_3; B_k)$ exists, and it is equal to 6, because the complete graph K_6 is B_k-free and $K_6 \rightarrow (K_3, K_3)^e$.

Problem 3.1. *Does the edge Folkman number $F_e(K_3, K_3; B_4)$ exist?*

In Theorem 5 we constructed a K_4-free and B_3-free graph G vertex arrowing $(3, 3)^v$. We think that it is an interesting challenge to solve the following graph existence problem for K_4-free and book-free graphs edge arrowing $(3, 3)^e$.

Problem 3.2. *For which $k \geq 4$ there exists a K_4-free and B_k-free graph G such that $G \rightarrow (3, 3)^e$?*

The answer seems not easy even just for $k = 4$. Note that a YES solution to Problem 3.1 does not provide an answer to Problem 3.2 with $k = 4$, while a NO answer to Problem 3.1 implies a NO answer to Problem 3.2 for $k = 4$. For Problem 3.2, we know that the answer is NO for $k = 3$ by Theorem 6 (hence we ask only about cases for $k \geq 4$), and clearly a YES answer for any k would imply YES answers for all $t > k$.

One of the most wanted Folkman numbers is $F_e(3, 3; 4) = F_e(K_3, K_3; K_4)$, for which the currently best known bounds are $20 \leq F_e(3, 3; 4)$ [1] and $F_e(3, 3; 4) \leq 786$ [7]. The value of $F_e(3, 3; 4)$ can be equivalently defined as the smallest number of vertices in any K_4-free graph which is not a union of two triangle-free graphs. An overview of what is known about this problem was presented in [16]. In particular, it was conjectured by Exoo that a special cubic residues $(4, 12)$-graph G_{127} on the vertex set $\mathcal{Z}_{127}$ is a witness to a much improved upper bound $F_e(3, 3; 4) \leq 127$, and likely its subgraphs may even give $F_e(3, 3; 4) \leq 94$ (see [16]). The graph G_{127} is K_4-free, has independence number 11, is B_{12}-free, but it contains a large number of subgraphs isomorphic to B_{11}. The Exoo's conjecture can be stated as $G_{127} \rightarrow (3, 3)^e$. If true, then it would give a YES answer in Problem 3.2 for all $k \geq 12$, leaving open the cases for $4 \leq k \leq 11$. Recall that by Theorem 6 the answer for $k = 3$ is NO.

4 More on arrowing triangles

In this section we study the existence of $F_e(K_3, K_3; H)$ for connected graphs H. First, we observe that, since graph avoidance is monotonic with respect to subgraphs, if a graph H is connected and $cl(H) \geq 4$, then there exist H-free graphs edge arrowing $(3,3)^e$, i.e. $F_e(K_3, K_3; H)$ exists, and obviously $F_e(K_3, K_3; H) \leq F_e(3,3;4)$. For 5 vertices, there are 4 such graphs, namely $\widehat{K}_{4,i}$ for $i \in [4]$, where $\widehat{K}_{n,s}$ is the graph obtained by connecting a new vertex v to s vertices of a K_n. Clearly, the numbers $F_e(K_3, K_3; \widehat{K}_{4,i})$ exist for $i \in [4]$, and $F_e(K_3, K_3; \widehat{K}_{4,i+1}) \leq F_e(K_3, K_3; \widehat{K}_{4,i})$ for $1 \leq i \leq 3$. In particular, note that $\widehat{K}_{4,3} = J_5$, $\widehat{K}_{4,4} = K_5$, and we have the easy bounds $15 = F_e(3,3;5) \leq F_e(K_3, K_3; J_5) \leq F_e(3,3;4) \leq 786$, using only what is known about $F_e(3,3;k)$ [16]. For $\widehat{K}_{4,i}$-free graphs, $i = 1, 2$, we have the following lemma.

Lemma 7. $F_e(K_3, K_3; \widehat{K}_{4,2}) = F_e(K_3, K_3; \widehat{K}_{4,1}) = F_e(3,3;4)$.

Proof. By the monotonicity of $F_e(K_3, K_3; \widehat{K}_{4,i})$ mentioned above, it is sufficient to prove that $F_e(K_3, K_3; \widehat{K}_{4,2}) \geq F_e(3,3;4)$. We will show that for any graph $G \in \mathcal{F}_e(K_3, K_3; \widehat{K}_{4,2})$ there exists a subgraph $G' \in \mathcal{F}_e(3,3;4)$ of G, which will complete the proof. Define graph G' on the same set of vertices as G, with the set of edges $E(G') = E(G) \setminus \{e \mid e \in K_4 \subset G\}$. Obviously, G' is K_4-free. Since G is $\widehat{K}_{4,2}$-free, we can see that every triangle in G which is not a triangle in G' has its three vertices in the same K_4 of G. Thus, any red-blue edge coloring of $E(G')$ without monochromatic triangles can be extended to whole $E(G)$ by independently red-blue coloring the edges of each K_4. This contradicts that $G \in \mathcal{F}_e(K_3, K_3; \widehat{K}_{4,2})$. Thus, no such coloring of $E(G')$ exists, and hence $G' \in \mathcal{F}_e(3,3;4)$. □

In the remainder of this section, we will consider only connected graphs H with K_3 but without K_4. There are three such graphs on 4 vertices, namely J_4 and its subgraphs, and hence as commented in Section 2, $F_e(K_3, K_3; H)$ does not exist in these cases. In the following, we focus attention on connected graphs H of order 5 with $cl(H) = 3$, and leave the study of such graphs with more than 5 vertices for future work. The next theorem claims the nonexistence of $F_e(K_3, K_3; H)$ for a special 5-vertex graph $H = K_1 + P_4$.

Theorem 8. *The edge Folkman number $F_e(K_3, K_3; K_1 + P_4)$ does not exist.*

Proof. The proof is very similar to that of Theorem 6. Suppose contrary, that $F_e(K_3, K_3; K_1 + P_4)$ exists, it is equal to n, and let G be any graph of order n in $\mathcal{F}_e(K_3, K_3; K_1 + P_4)$. For any vertex $u \in V(G)$ we must have $G - u \not\rightarrow (K_3, K_3)^e$. Fix any vertex $u \in V(G)$, and let H be the graph induced in G by the neighbors of u, $H = G[N(u)]$. Since G is $(K_1 + P_4)$-free, H does not contain P_4. Therefore any connected component of H is bipartite or isomorphic to K_3. Now, the same steps as in the proof of Theorem 6 lead to a contradiction. □

We now state a theorem summarizing the existence of $F_e(K_3, K_3; H)$ for all connected graphs H on 5 vertices with $cl(H) = 3$. Only two cases remain open, namely those for the wheel graph W_5 and the complement of $P_2 \cup P_3$. These cases should be studied more, and we expect that new insights can be important for better understanding of which graphs edge arrow $(3,3)^e$.

Theorem 9. *Let H be any connected K_4-free graph on 5-vertices containing K_3. Then the edge Folkman number $F_e(K_3, K_3; H)$ does not exist, except for two possible cases for H, namely W_5 and $\overline{P_2 \cup P_3}$.*

Proof. There are 11 nonisomorphic K_4-free connected graphs on 5 vertices containing K_3. By Theorem 8, $F_e(K_3, K_3; K_1 + P_4)$ does not exist. The graph $K_1 + P_4$ contains as a subgraph 7 further such graphs H (including the bowtie graph $K_1 + 2K_2$, $K_{1,4} + e$, and the so-called bull graph), for which by monotonicity $F_e(K_3, K_3; H)$ does not exist either. This leaves three cases: B_3, W_5 and $\overline{P_2 \cup P_3}$. The first case was eliminated by Theorem 6, while the other two are as the stated exceptions. □

Problem 4.1. *Prove or disprove the existence*
(a) *of the edge Folkman number $F_e(K_3, K_3; \overline{P_2 \cup P_3})$, and*
(b) *of the edge Folkman number $F_e(K_3, K_3; K_1 + C_4)$.*

Note that $W_5 = K_1+C_4$ is a subgraph of $J_5 = K_5-e$. Hence, if $F_e(K_3, K_3; W_5)$ exists, then we have $F_e(K_3, K_3; J_5) \leq F_e(K_3, K_3; W_5)$. The analogous statement holds for the complement of $P_2 \cup P_3$. On the other hand, the latter is a subgraph of W_5, hence there are only three possible combined YES/NO answers to the existence questions (a) and (b) in Problem 4.1, namely NO/NO, YES/YES and NO/YES.

A natural direction to generalize considerations of this section is to analyze which small graphs on at least 6 vertices necessarily are subgraphs of every K_4-free graph edge arrowing $(3,3)^e$. The simplest candidate for such a graph is $B_4 = K_6 \setminus K_4$, as stated in Problem 3.1. One could also proceed by making a catalog of small subgraphs in known witnesses of existence of $F_e(3,3;4)$, in particular for the graph G_{786}, which currently is the smallest known such graph [7]. This, and even only some conditional answers to our problems, may lead to better bounds on $F_e(3,3;4)$.

5 Some cases of multicolor Ramsey arrowing

Since we know that $F_e(K_3, K_3; J_4)$ does not exist, if a 3-color edge arrowing $G \to (K_3, K_3, K_k)^e$ holds, then we must have $G \to (J_4, K_k)^e$. This easily generalizes to $F_e(K_3, K_3, K_k; K_s) \geq F_e(J_4, K_k; K_s)$ for $s > k \geq 3$, and in particular it gives $F_e(3,3,3;4) \geq F_e(J_4, K_3; K_4)$. We note that $F_e(3,3,3;4)$ exists, its value is unknown, it is likely quite large, and probably still much harder to obtain than the notoriously difficult case of $F_e(3,3;4)$. Clearly, the same reasoning holds for any graph H instead of J_4 for which $F_e(K_3, K_3; H)$ does not exist, including B_3, $K_1 + P_4$ or other graphs discussed in the previous section. This leads to the following corollary.

Corollary 10. *If H is any graph for which $F_e(K_3, K_3; H)$ does not exist, then for $s > k \geq 3$ we have*

$$F_e(3, 3, k; s) \geq F_e(H, K_k; K_s).$$

Proof. As in the comments above, we observe that any n-vertex graph G witnessing the upper bound $F_e(3, 3, k; s) \leq n$ must also satisfy $G \rightarrow (H, K_k)^e$. Thus we have $F_e(H, K_k; K_s) \leq n$. □

It would be interesting to construct a K_4-free graph G such that $G \rightarrow (K_3, J_4)^e$ but $G \not\rightarrow (3, 3, 3)^e$. This might be quite hard since it is difficult to construct any K_4-free graph that arrows $(K_3, J_4)^e$, and it would be another challenge to show that it does not arrow $(3, 3, 3)^e$. Similarly, obtaining any nontrivial lower bound for the difference $F_e(3, 3, 3; 4) - F_e(K_3, J_4; K_4)$ seems difficult.

On the other hand, there exists an interesting example of a K_4-free graph G on 30193 vertices, constructed by Lu [8], such that $G \rightarrow (J_4, J_4)^e$ (thus also $G \rightarrow (K_3, J_4)^e$). It is possible that for this graph we have $G \rightarrow (3, 3, 3)^e$, however we do not know how to prove or disprove the latter. Also, note that by an argument as in the proof of Corollary 10 we have $F_e^4(3; 4) = F_e(3, 3, 3, 3; 4) \geq F_e(J_4, J_4; K_4)$.

Finally, we establish a new link between some two-color edge Folkman numbers and multicolor vertex Folkman numbers. They generalize a result obtained in [17].

Lemma 11. *For $k \geq s \geq 2$ and graphs G and H, if G is H-free, $H \subset K_{k+1}$, and $G \rightarrow (K_s, K_k)^e$, then for every vertex $u \in V(G)$ and $s-1$ colors we have $G - u \rightarrow (K_k, \cdots, K_k)^v$.*

Proof. For a contradiction, suppose that for some graphs G and H as specified in the lemma, and for some vertex $u \in V(G)$, there exists a partition $V(G-u) = \bigcup_{i=1}^{s-1} V_i$, such that the graphs $G[V_i]$ are K_k-free, for every $i \in [s-1]$.

Now, we color red or blue all the edges in $E(G)$ as follows. All edges in each $G[V_i]$, for $i \in [s-1]$, are colored blue. The edges in $G[N(u)]$ are also blue. The edges between u and $N(u)$ are red, and all other edges in $E(G)$, which are necessarily between different parts V_i, are also colored red. Note that any red clique may have at most one vertex in each of the parts V_i, and that there are no red triangles passing through vertex u. Thus, this coloring has no red K_s. No nontrivial blue clique contains vertex u, and none of $G[V_i]$ contains blue K_k, hence any potential blue K_k on vertices S must intersect different parts V_i. However, if such S exists, and because of how the coloring was defined, the set of vertices $S \cup \{u\}$ would form a K_{k+1}, contrary to the assumption that G is H-free. □

Corollary 12. *For $2 \leq s \leq k$ and graph $H \subset K_{k+1}$, if $F_e(K_s, K_k; H)$ exists, then $F_v^{s-1}(K_k; H)$ also exists and $F_e(K_s, K_k; H) \geq F_v^{s-1}(K_k; H) + 1$.*

Proof. Consider any graph G, such that $G \in \mathcal{F}_e(K_s, K_k; H)$, of the least possible order $F_e(K_s, K_k; H)$. Then by Lemma 11, the graph $G - u$ is in the set $\mathcal{F}_v^{s-1}(K_k; H)$ and it has one vertex less than G. This proves the inequality. $\square$

The proofs of our last lemma and corollary use a method similar to one applied in the proof of $F_e(3, k; k+1) > F_v(k, k; k+1)$ in [17]. The latter is a special case of Corollary 12 with $s = 3$ and $H = K_{k+1}$. Another interesting instantiation of Corollary 12 is for $H = J_{k+1}$, for which the existence question of corresponding Folkman numbers was discussed in Section 2.

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