On a Diagonal Conjecture for Classical Ramsey Numbers

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Abstract

Let $R(k_1, \cdots, k_r)$ denote the classical r-color Ramsey number for integers $k_i \geq 2$. The Diagonal Conjecture (DC) for classical Ramsey numbers poses that if $k_1, \cdots, k_r$ are integers no smaller than 3 and $k_{r-1} \leq k_r$, then $R(k_1, \cdots, k_{r-2}, k_{r-1} - 1, k_r + 1) \leq R(k_1, \cdots, k_r)$. We obtain some implications of this conjecture, present evidence for its validity, and discuss related problems.

Let $R_r(k)$ stand for the r-color Ramsey number $R(k, \cdots, k)$. It is known that $\lim_{r\to\infty} R_r(3)^{1/r}$ exists, either finite or infinite, the latter conjectured by Erdős. This limit is related to the Shannon capacity of complements of K_3-free graphs. We prove that if DC holds, and $\lim_{r\to\infty} R_r(3)^{1/r}$ is finite, then $\lim_{r\to\infty} R_r(k)^{1/r}$ is finite for every integer $k \geq 3$.

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1 Introduction

Denote by K_n the complete graph on n vertices. The classical multicolor Ramsey number $R(k_1, \cdots, k_r)$ is the smallest positive integer n such that if we color the edges of K_n with r colors, then in this coloring there must be a monochromatic K_{k_i} whose all edges are in color i, for some $i \in \{1, \ldots, r\}$. In the diagonal case $k = k_1 = \cdots = k_r$ we will use the simpler notation $R_r(k) = R(k_1, \cdots, k_r)$.

Wang Rui [11] in a 2008 paper claimed to prove that in the two-color case it holds that $R(s,t) > R(s-1, t+1)$ for $s \leq t$. Or, equivalently, one of his theorems states that as we move away from the diagonal of the table with Ramsey numbers $R(s,t)$, while preserving $s+t$, the values decrease. Known values and bounds for Ramsey numbers [10] do not contradict this claim, and actually, it seems very plausible to be true. Unfortunately, it is rather evident that its proof in [11] is not correct. The problems with this paper are numerous, starting with a strange alternate definition of Ramsey numbers, followed by unfounded circular arguments between the alternate definitions. Wang in his paper is addressing almost exclusively two-color cases, but towards the end he also makes some claims for more colors, though again without what can be considered rigorous proofs.

We summarize the above in the following conjecture for general multicolor Ramsey numbers, where two colors are a special case.

Diagonal Conjecture (DC).
If $k_1, \cdots, k_r$ are integers no smaller than 3, $r \geq 2$, and $k_{r-1} \leq k_r$, then

$$R(k_1, \cdots, k_{r-2}, k_{r-1} - 1, k_r + 1) \leq R(k_1, \cdots, k_r).$$

If DC holds, then for the last two colors (and thus also for any two fixed colors) as we move away from the diagonal, while preserving $k_{r-1} + k_r$, the corresponding Ramsey number cannot increase. We believe that a stronger version of DC with $<$ instead of $\leq$ also holds. Still, even the weaker version can be very hard to prove.

In 1983, Chung and Grinstead [6] showed that $\lim_{r\to\infty} R_r(3)^{1/r}$ exists, though it is not known whether this limit is finite or infinite. The same argument can be used to show that $\lim_{r\to\infty} R_r(k)^{1/r}$ also exists for all $k > 3$, again finite or infinite. Erdős was inclined to think that $\lim_{r\to\infty} R_r(3)^{1/r} = \infty$ (cf. [9, 13]). This limit is also closely related to the Shannon capacity of complements of K_3-free graphs (i.e. graphs with independence number equal to 2), which was discussed in an earlier paper by the second and third authors [12].

Let $L_k = \lim_{r\to\infty} R_r(k)^{1/r}$. By monotonicity of Ramsey numbers, we can easily see that $L_{k+1} \geq L_k$ for all $k \geq 3$, including the propagation of infinity to larger indices. In this paper we obtain some consequences of the assumption that the DC holds, we present evidence for its validity, and discuss related problems. In particular, we prove that if DC holds and $\lim_{r\to\infty} R_r(3)^{1/r}$ is finite, then $\lim_{r\to\infty} R_r(k)^{1/r}$ is finite for any integer $k \geq 3$. We also discuss other relationships between DC and the sequence of L_k's.

2 Some Consequences of DC

Lemma 1. *If DC holds, then for every integer $k \geq 3$ we have*

$$R_{2r}(k) - 1 \geq (R_r(k-1) - 1)(R_r(k+1) - 1).$$

Proof. An old result obtained by Abbott [1], also presented in [16] (Theorem 2, page 7), states that if $k_j \geq 2$ for $1 \leq j \leq r$, then for all $1 < i < r$ we have

$$R(k_1, \cdots, k_r) > (R(k_1, \cdots, k_i) - 1)(R(k_{i+1}, \cdots, k_r) - 1). \tag{1}$$

If DC holds, then we can apply it r times to $R_{2r}(k)$ to obtain

$$R_{2r}(k) - 1 \geq R(k-1, \cdots, k-1, k+1, \cdots, k+1) - 1.$$

Now, we can complete the proof using inequality (1). □

Theorem 2. *If DC holds and $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$ is finite, then $\lim_{r\to\infty} R_r(k)^{\frac{1}{r}}$ is finite too, for every integer $k \geq 3$.*

Proof. For every integer $k \geq 3$, using Lemma 1 with DC, we have

$$(R_{2r}(k) - 1)^{1/r} \geq (R_r(k-1) - 1)^{1/r}(R_r(k+1) - 1)^{1/r},$$

and thus

$$\frac{(R_{2r}(k) - 1)^{\frac{1}{2r}}}{(R_r(k-1) - 1)^{1/r}} \geq \frac{(R_r(k+1) - 1)^{1/r}}{(R_{2r}(k) - 1)^{\frac{1}{2r}}}. \tag{2}$$

Clearly, $\lim_{r\to\infty} R_r(i)^{1/r} = \lim_{r\to\infty}(R_r(i) - 1)^{1/r}$ for $i \in \{k-1, k, k+1\}$. Taking it into account in inequality (2) leads to

$$\lim_{r\to\infty} \frac{R_r(k)^{\frac{1}{r}}}{R_r(k-1)^{\frac{1}{r}}} \geq \lim_{r\to\infty} \frac{R_r(k+1)^{\frac{1}{r}}}{R_r(k)^{\frac{1}{r}}}. \tag{3}$$

Note that $R_r(2) = 2$. Finally, we can prove the claim of the theorem by induction on k. The base case is for $k = 3$, which is the finiteness of $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$. The inductive step follows from the inequality (3). □

By Theorem 2, we can see that if DC holds, then either $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}} = \infty$, or $\lim_{r\to\infty} R_r(k)^{\frac{1}{r}}$ is finite for every $k \geq 3$. Or, equivalently, DC and $\lim_{r\to\infty} R_r(k)^{\frac{1}{r}} = \infty$ for any $k \geq 3$ implies that $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}} = \infty$. On the other hand, if $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$ were finite, then it would support our intuition that the best known lower bounds for $R_r(3)$ are much closer to the exact values than the currently best known upper bounds.

Table 1 presents the best known lower and upper bounds on $R_r(3)$ for $r \leq 10$. The exact values for $r = 2, 3$ are known, and it was conjectured that $R_4(3) = 51$, i.e. that the current lower bound for $r = 4$ is equal to the exact value [13]. Lower bounds for higher r in Table 1 are implied by sum-free set constructions and related Schur numbers (cf. [16, 13]), in particular they imply that $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}} \geq 1073^{\frac{1}{6}} \approx 3.1996$.

For the upper bound, a simple reasoning yields $R_r(3) \leq 3r!$, while the best known general upper is just a little better, namely, the third author et al. proved that for $r \geq 4$ we have the bound $R_r(3) \leq (e - \frac{1}{6})r! + 1 \approx 2.55r!$ [15]. The latter was proved based on the bound $R_4(3) \leq 62$, which in turn was obtained with the help of significant computations. This is the only case where we know of an upper bound for a Ramsey number of this form that is better than one obtained by simple steps using smaller cases. Complete references to lower and upper bounds and other general results on $R_r(k)$ can be found in the dynamic survey paper by the second author [10].

If our perspective above that the lower bounds in Table 1 are much closer to $R_r(k)$ than the upper bounds is correct, it would add weight to the case of $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$ being finite, and thus by DC and Theorem 2 also that all limits $\lim_{r\to\infty} R_r(k)^{\frac{1}{r}}$ are finite.

r	lower bound	upper bound
2	**6**	6
3	**17**	17
4	**51**	62
5	162	307
6	538	1838
7	1682	12861
8	5204	102882
9	16146	925931
10	51202	9259302

Table 1. Known bounds on $R_r(3)$ for $r \leq 10$.

We can prove the following Theorem 3 about the growth of the limits $\lim_{r\to\infty} R_r(k)^{\frac{1}{r}}$ with increasing k, only assuming that DC holds. However, we feel strongly that it also holds unconditionally.

Theorem 3. *If DC holds, then for every integer $k \geq 3$, we have*

$$\lim_{r\to\infty} \frac{R_r(k)^{\frac{1}{r}}}{R_r(k-1)^{\frac{1}{r}}} > 1.$$

Proof. Consider a general constructive lower bound for multicolor Ramsey numbers $R_r(st+1) > (R_r(s+1)-1)(R_r(t+1)-1)$, which can be obtained from a standard graph product construction as described in [16] (inequality (5) on page 4 there). Using $s = k-1$ and $t = 2$, it gives $R_r(2k-1) > (R_r(3)-1)(R_r(k)-1)$. We know that asymptotically $R_r(3)$ grows at least as fast as 3.19^r, but one can also easily observe that $R_r(3) > 2^r$ holds for all r. Thus

$$\frac{R_r(2k-1)-1}{R_r(k)-1} \geq R_r(3) - 1 \geq 2^r,$$

and hence

$$\lim_{r\to\infty} \frac{R_r(2k)^{\frac{1}{r}}}{R_r(k)^{\frac{1}{r}}} \geq 2. \tag{4}$$

Assume that DC holds, and for contradiction suppose that for some $a \geq 3$ we have $\lim_{r\to\infty} R_r(a)^{\frac{1}{r}}/R_r(a-1)^{\frac{1}{r}} = 1$. Note that the inequality (3) in the proof of Theorem 2 is a consequence of just DC, and it is valid for any $a \geq 3$. Consequently, using (3) we can conclude that

$$\lim_{r\to\infty} \frac{R_r(k)^{\frac{1}{r}}}{R_r(k-1)^{\frac{1}{r}}} = 1$$

for any integer $k \geq a$. This, however, leads to

$$\lim_{r\to\infty} \frac{R_r(2a)^{\frac{1}{r}}}{R_r(a)^{\frac{1}{r}}} = \lim_{r\to\infty} \prod_{k=a+1}^{2a} \frac{R_r(k)^{\frac{1}{r}}}{R_r(k-1)^{\frac{1}{r}}} = 1,$$

which contradicts (4). This completes the proof of the theorem. □

Corollary 4. *For $k \geq 3$, let $L_k = \lim_{r\to\infty} R_r(k)^{1/r}$, and assume that DC holds. Then it is true that:*

(a) all L_k's are finite or all of them are infinite, and
(b) if L_3 is finite then $L_k < L_{k+1}$ for all $k \geq 3$.

Proof. As discussed in the Introduction, all the limits L_k exist and they satisfy $L_k \leq L_{k+1}$, regardless of whether DC holds or not. Thus, the claim (a) follows from Theorem 2 and claim (b) follows from Theorem 3. □

We wish to note that clearly $\lim_{k\to\infty} L_k$ is infinite, even without assuming validity of DC. This can be seen using an easy bound $R_r(k) > (k-1)^r$ implied by results obtained by Abbott [1] (cf. (7) and (4) in [16]).

Observe an obvious equivalence that $R(s,t) \geq R(s-1,t+1)$ if and only if $R(s,t)-R(s-1,t) \geq R(s-1,t+1)-R(s-1,t)$, which for $3 \leq s \leq t$ can be seen as just another way of looking at the DC. It might seem that the analysis of how $R(s,t)-R(s-1,t)$ relates to $R(s-1,t+1)-R(s-1,t)$ should be simpler, but it apparently resists to be so. Some related discussion can be found in [5, 14, 17].

3 Current Evidence for DC

This section presents some additional observations which make us believe that DC holds. We note that Wang Rui [11] did not provide much intuition behind the conjecture itself, perhaps because he thought that he had proved it as a theorem. If so, then more discussion would not be required.

Below, we split our comments into two cases: of two colors and of more colors. Only just a few exact values of Ramsey numbers are known, hence not many absolute instances confirming the DC can be pointed to. On the other hand, for a large number of open cases, say such as $R(s,t)$ for specific s and t, it seems that the best known lower bound is much closer to the exact value than known upper bound. Historically (see the past revisions of [10]), the lower bounds often slowly improve over some time then stabilize, while the upper bounds are improved rarely and most of the time only with a large computational effort. Or, in other words, known upper bounds are far from being tight because we know very little about how to improve them. Thus, similarly as in the previous section when arguing for the finiteness of $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$, our evidence will rely greatly on what we know about lower bounds.

Two Colors

Let $DC(s,t)$ stand for the validity of $R(s,t) \geq R(s-1,t+1)$. We will consider various $DC(s,t)$ statements for special values of the parameters s and t, but always satisfying $3 \leq s \leq t$.

(a) $DC(3,t)$ is true, since easily $R(3,t) > R(2,t+1) = t+1$ for all $t \geq 3$.

(b) $DC(4,t)$ is true, since we have $R(4,t) \geq R(3,t) + 2t - 3$ ([5], see also [17]) but easily $R(3,t+1) \leq R(3,t) + t + 1$ for all $t \geq 4$.

(c) $DC(5,5)$ is true, since it is known that $R(5,5) \geq 43$ and $R(4,6) \leq 41$ (cf. [10]). Angeltveit and McKay in a recent unpublished project [8] obtained the upper bounds $R(4,7) \leq 58$ and $R(4,8) \leq 79$, which confirm the validity of $DC(5,6)$ and $DC(5,7)$ by using previously published lower bounds $R(5,6) \geq 58$ and $R(5,7) \geq 80$ (cf. [10]).

(d) The above establishes the validity of $DC(s,t)$ for all $s < 5$ and all cases with $s+t \leq 12$, except $DC(6,6)$. For the latter it is known that $R(6,6) \geq 102$ and $R(5,7) \leq 143$, though recall our previous comments, and especially in this case we feel that the lower bound is strong but the upper bound very weak.

In one larger case, namely that of $DC(8,10)$, the best known lower bounds $R(8,10) \geq 343$ and $R(7,11) \geq 405$ (cf. [10]) do not "follow" the DC (but do not contradict it either). We believe that this is because of a rather special construction establishing the bound for $R(7,11)$, while the bound for $R(8,10)$ was obtained by a heuristic search restricted to only circular graphs. This suggests that it should be feasible to significantly improve the current lower bound for $R(8,10)$.

We also note that known bounds for $R(s,t)$ collected in [10] do not contradict $DC(s,t)$ for any $3 \leq s \leq t$.

(e) The further we go from the diagonal of DC, the easier it seems to corroborate it. We anticipate this problem to be the hardest on the diagonal itself, i.e. proving that $R(t,t) \geq R(t-1,t+1)$ for any $t \geq 6$.

(f) In 2010, Bohman and Keevash [4] proved that for fixed $s \geq 5$ and $t \to \infty$ we have the following lower bound

$$R(s,t) = \Omega(t^{\frac{s+1}{2}}(\log s)^{\frac{1}{s-2}-\frac{s+1}{2}}).$$

This result does not resolve any concrete $DC(s,t)$ instances, yet, again using our perspective on lower bounds, builds up evidence for the validity of $DC(s,t)$ for fixed s and large t.

More Colors

In the multicolor cases, almost all evidence we have for DC is based on lower bounds, even more so than in the case of two colors. Table 2 lists 11 pairs of parameters (P_1, P_2) together with the corresponding best known lower bounds (LB_1, LB_2) listed in [10] for $R(k_1, \cdots, k_{r-2}, k_{r-1}-1, k_r+1)$ and $R(k_1, \cdots, k_r)$, with $4 \leq k_{r-1} \leq k_r$. This includes essentially all evidence of this type we have for $k_{r-1} \geq 4$.

P_1	LB_1	LB_2	P_2
3,3,5	45	55	3,4,4
3,3,6	61	89	3,4,5
3,3,7	85	117	3,4,6
3,3,8	103	152	3,4,7
3,3,9	129	193	3,4,8
3,3,10	150	242	3,4,9
3,4,6	117	139	3,5,5
3,4,7	152	181	3,5,6
3,4,8	193	241	3,5,7
4,3,5	89	128	4,4,4
3,3,3,5	162	171	3,3,4,4

Table 2. The best known lower bounds LB_1 and LB_2 on Ramsey numbers $R(k_1, \cdots, k_{r-2}, k_{r-1} - 1, k_r + 1)$ and $R(k_1, \cdots, k_r)$, for some DC-adjacent pairs (P_1, P_2), where $P_1 = (k_1, \cdots, k_{r-2}, k_{r-1} - 1, k_r + 1)$ and $P_2 = (k_1, \cdots, k_r)$.

We can say a little more beyond Table 2 for some combinations of parameters in P_2 involving $k_{r-1} = 3$. For example, we clearly have $R(5, k) = R(k, 2, 5)$, and by inspection of bounds reported in [10], we can see that $R(k, 3, 4) \geq R(k, 2, 5)$ holds for $2 \leq k \leq 7$.

The lower bounds in columns LB_1 and LB_2 do get occasional improvements, though not often and not by much. For a particular P_1 to contradict DC, the corresponding lower bound LB_1 would have to exceed not only LB_2, but also its associated upper bound.

4 Some Problems Related to DC and $R_r(k)$

(1) For connected graphs $G_1, \cdots, G_r$, the generalized multicolor Ramsey number $R(G_1, \cdots, G_r)$ is defined as the smallest integer n such that in any r-coloring of the edges of K_n there must be a monochromatic G_i in color i, for some $1 \leq i \leq r$. We pose the following question generalizing DC. For $G_{r-1} = K_s$, $G_r = K_t$ with $s \leq t$, is it true that

$$R(G_1, G_2, \cdots, K_{s-1}, K_{t+1}) \leq R(G_1, G_2, \cdots, K_s, K_t)?$$

We think that it is true, but stop here and do not make it another conjecture.

(2) Let $r \geq 3$, $k_i \geq 3$, and $k_{r-1} \leq k_r$. Suppose that C is a coloring of the edges of K_n witnessing the lower bound $n < R(k_1, \cdots, k_{r-2}, k_{r-1} - 1, k_r + 1)$. Define the graph G to consist of the edges of C in colors $r - 1$ and r. Is it true that $G \not\rightarrow (k_{r-1}, k_r)^e$, i.e. that there exists a 2-coloring of the edges of G without any monochromatic $K_{k_{r-1}}$ in the first color and K_{k_r} in the second color? We think that the answer is YES, but less strongly than in (1).

(3) The Shannon capacity of a noisy channel modeled by graph G, often referred to as the Shannon capacity of G, is defined as the limit

$$c(G) = \lim_{r\to\infty} \alpha(G^r)^{\frac{1}{r}},$$

where $\alpha(G^r)$ is the independence number of the strong r-th power of G. The capacity $c(G)$ measures the efficiency of the best possible strategy when sending long words over a noisy channel modeled by G. It was studied extensively in information theory by many authors, including [2, 3]. In a very short 1971 paper, Erdős et al. [7] proved that for each k there exists a graph G with $\alpha(G) = k$ such that $\alpha(G^r) + 1 = R_r(k + 1)$. This provides an implicit link between Shannon capacity and Ramsey numbers, and in particular to the problem of finiteness of the limit $\lim_{r\to\infty} R_r(k)^{1/r}$. We explored it further in [12], where we proved that $\lim_{n\to\infty} R_r(3)^{1/r}$ is the supremum of the Shannon capacity of complements of K_3-free graphs but it cannot be achieved by any finite graph power. In general, for any fixed integer $k \geq 3$, we have that $\lim_{r\to\infty} R_r(k)^{1/r}$ is equal to the supremum of the Shannon capacity $c(G)$ over all graphs G with independence number $k - 1$, but this supremum cannot be achieved by any finite graph power either.

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