



Integration By Parts

Formula

$$\int u dv = uv - \int v du$$

I. Guidelines for Selecting u and dv:

(There are always exceptions, but these are generally helpful.)

“L-I-A-T-E” Choose ‘u’ to be the function that comes first in this list:

L: Logarithmic Function

I: Inverse Trig Function

A: Algebraic Function

T: Trig Function

E: Exponential Function

Example A: $\int x^3 \ln x \, dx$

*Since $\ln x$ is a logarithmic function and x^3 is an algebraic function, let:

$$u = \ln x \quad (\text{L comes before A in LIATE})$$

$$dv = x^3 \, dx$$

$$du = \frac{1}{x} \, dx$$

$$v = \int x^3 \, dx = \frac{x^4}{4}$$

$$\int x^3 \ln x \, dx = uv - \int v \, du$$

$$= (\ln x) \frac{x^4}{4} - \int \frac{x^4}{4} \frac{1}{x} \, dx$$

$$= (\ln x) \frac{x^4}{4} - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} (\ln x) - \frac{1}{4} \frac{x^4}{4} + C$$

$$= \frac{x^4}{4} (\ln x) - \frac{x^4}{16} + C \quad \text{ANSWER}$$

Example B: $\int \sin x \ln(\cos x) dx$

$$u = \ln(\cos x) \text{ (Logarithmic Function)}$$

$$dv = \sin x dx \text{ (Trig Function [L comes before T in LIATE])}$$

$$du = \frac{1}{\cos x}(-\sin x) dx = -\tan x dx$$

$$v = \int \sin x dx = -\cos x$$

$$\begin{aligned}\int \sin x \ln(\cos x) dx &= uv - \int v du \\ &= (\ln(\cos x))(-\cos x) - \int (-\cos x)(-\tan x) dx \\ &= -\cos x \ln(\cos x) - \int (\cos x) \frac{\sin x}{\cos x} dx \\ &= -\cos x \ln(\cos x) - \int \sin x dx \\ &= -\cos x \ln(\cos x) + \cos x + C \quad \text{ANSWER}\end{aligned}$$

Example C: $\int \sin^{-1} x dx$

*At first it appears that integration by parts does not apply, but let:

$$u = \sin^{-1} x \text{ (Inverse Trig Function)}$$

$$dv = 1 dx \text{ (Algebraic Function)}$$

$$du = \frac{1}{\sqrt{1-x^2}} dx$$

$$v = \int 1 dx = x$$

$$\begin{aligned}\int \sin^{-1} x dx &= uv - \int v du \\ &= (\sin^{-1} x)(x) - \int x \frac{1}{\sqrt{1-x^2}} dx \\ &= x \sin^{-1} x - \left(-\frac{1}{2}\right) \int (1-x^2)^{-1/2} (-2x) dx \\ &= x \sin^{-1} x + \frac{1}{2} (1-x^2)^{1/2} (2) + C \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C \quad \text{ANSWER}\end{aligned}$$

II. Alternative General Guidelines for Choosing u and dv :

- A. Let dv be the most complicated portion of the integrand that can be “easily” integrated.
- B. Let u be that portion of the integrand whose derivative du is a “simpler” function than u itself.

Example: $\int x^3 \sqrt{4-x^2} dx$

*Since both of these are algebraic functions, the LIATE Rule of Thumb is not helpful. Applying Part (A) of the alternative guidelines above, we see that $x\sqrt{4-x^2}$ is the “most complicated part of the integrand that can easily be integrated.” Therefore:

$$dv = x\sqrt{4-x^2} dx$$

$$u = x^2 \quad (\text{remaining factor in integrand})$$

$$du = 2x dx$$

$$v = \int x\sqrt{4-x^2} dx = -\frac{1}{2} \int (-2x)(4-x^2)^{1/2} dx$$

$$= \left(-\frac{1}{2}\right)\left(\frac{2}{3}\right) (4-x^2)^{3/2} = -\frac{1}{3}(4-x^2)^{3/2}$$

$$\int x^3 \sqrt{4-x^2} dx = uv - \int v du$$

$$= (x^2)\left(-\frac{1}{3}(4-x^2)^{3/2}\right) - \int -\frac{1}{3}(4-x^2)^{3/2}(2x) dx$$

$$= \frac{-x^2}{3}(4-x^2)^{3/2} - \frac{1}{3} \int (4-x^2)^{3/2}(-2x) dx$$

$$= \frac{-x^2}{3}(4-x^2)^{3/2} - \frac{1}{3}(4-x^2)^{5/2}\left(\frac{2}{5}\right) + C$$

$$= \frac{-x^2}{3}(4-x^2)^{3/2} - \frac{2}{15}(4-x^2)^{5/2} + C \quad \mathbf{Answer}$$

III. Using repeated Applications of Integration by Parts:

Sometimes integration by parts must be repeated to obtain an answer.

Note: DO NOT switch choices for u and dv in successive applications.

Example: $\int x^2 \sin x \, dx$

$$u = x^2 \text{ (Algebraic Function)}$$

$$dv = \sin x \, dx \text{ (Trig Function)}$$

$$du = 2x \, dx$$

$$v = \int \sin x \, dx = -\cos x$$

$$\begin{aligned} \int x^2 \sin x \, dx &= uv - \int v du \\ &= x^2(-\cos x) - \int -\cos x \, 2x \, dx \\ &= -x^2 \cos x + 2 \int x \cos x \, dx \end{aligned}$$

Second application of integration by parts:

$$u = x \text{ (Algebraic function) (Making "same" choices for } u \text{ and } dv)$$

$$dv = \cos x \text{ (Trig function)}$$

$$du = dx$$

$$v = \int \cos x \, dx = \sin x$$

$$\begin{aligned} \int x^2 \sin x \, dx &= -x^2 \cos x + 2 [uv - \int v du] \\ &= -x^2 \cos x + 2 [x \sin x - \int \sin x \, dx] \\ &= -x^2 \cos x + 2 [x \sin x + \cos x + c] \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c \quad \mathbf{Answer} \end{aligned}$$

Note: After each application of integration by parts, watch for the appearance of a constant multiple of the original integral.

Example: $\int e^x \cos x \, dx$

$$u = \cos x \quad (\text{Trig function})$$

$$dv = e^x \, dx \quad (\text{Exponential function})$$

$$du = -\sin x \, dx$$

$$v = \int e^x \, dx = e^x$$

$$\begin{aligned} \int e^x \cos x \, dx &= uv - \int v du \\ &= \cos x \, e^x - \int e^x (-\sin x) \, dx \\ &= \cos x \, e^x + \int e^x \sin x \, dx \end{aligned}$$

Second application of integration by parts:

$$u = \sin x \quad (\text{Trig function}) \quad (\text{Making "same" choices for } u \text{ and } dv)$$

$$dv = e^x \, dx \quad (\text{Exponential function})$$

$$du = \cos x \, dx$$

$$v = \int e^x \, dx = e^x$$

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \cos x + (uv - \int v du) \\ \int e^x \cos x \, dx &= e^x \cos x + \sin x \, e^x - \int e^x \cos x \, dx \end{aligned}$$

Note appearance of original integral on right side of equation. Move to left side and solve for integral as follows:

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2}(e^x \cos x + e^x \sin x) + C \quad \textbf{Answer}$$

Practice Problems:

1. $\int 3x e^{-x} dx$

2. $\int \frac{\ln x}{x^2} dx$

3. $\int x^2 \cos x dx$

4. $\int x \sin x \cos x dx$

5. $\int \cos^{-1} x dx$

6. $\int (\ln x)^2 dx$

7. $\int x^3 \sqrt{9-x^2} dx$

8. $\int e^{2x} \sin x dx$

9. $\int x^2 \sqrt{x-1} dx$

10. $\int \frac{1}{x(\ln x)^3} dx$

$$1. \quad -3xe^{-x} - 3e^{-x} + C$$

$$u = 3x$$

$$dv = e^{-x} dx$$

$$2. \quad -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$u = \ln x$$

$$dv = \frac{1}{x^2} dx$$

$$3. \quad x^2 \sin x + 2x \cos x - 2 \sin x + C$$

$$u = x^2$$

$$dv = \cos x \, dx$$

$$4. \quad -\frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + C$$

$$\text{note: } \frac{\sin 2x}{2} = \sin x \cos x$$

$$u = x$$

$$dv = \sin 2x \cos x \, dx$$

$$5. \quad x \cos^{-1} x - \sqrt{1-x^2} + C$$

$$u = \cos^{-1} x$$

$$dv = dx$$

$$6. \quad x(\ln x)^2 - 2x \ln x + 2x + C$$

$$u = (\ln x)^2$$

$$dv = dx$$

$$7. \quad -\frac{x^2}{3}(9-x^2)^{3/2} - \frac{2}{15}(9-x^2)^{5/2} + C$$

$$u = x^2$$

$$dv = (4-x^2)^{1/2} x \, dx$$

$$8. \quad \frac{2e^{2x} \sin x}{5} - \frac{e^{2x} \cos x}{5} + C$$

$$u = \sin x$$

$$dv = e^{2x} dx$$

$$9. \quad \frac{2x^2(x-1)^{3/2}}{3} - \frac{8x(x-1)^{5/2}}{15} + \frac{16(x-1)^{7/2}}{105} + C \quad u = x^2$$

$$dv = (x-1)^{1/2} dx$$

$$10. \quad \frac{-1}{2(\ln x)^2} + C$$

$$u = \frac{1}{(\ln x)^3} = (\ln x)^{-3}$$

$$dv = \frac{1}{x} dx$$