Partial Fractions

A rational function, i.e. a quotient of polynomials: $f(x)=\dfrac{P(x)}{Q(x)}$ can be expressed as a sum of simpler fractions, called **partial fractions**.

This technique is used for rewriting problems so that they can be integrated. For example, the integral $\int \dfrac{x+7}{x^2-x-6}\,dx$ can be rewritten as $\int\left(\dfrac{2}{x-3}-\dfrac{1}{x+2}\right)dx$ using the method of partial fractions. This is then easily integrated as $2\ln|x-3|-\ln|x+2|+C$.

1. If you have an **improper rational function** (the degree of the numerator is <u>equal to or greater than</u> the degree of the denominator) the preliminary step of long division is necessary: $Q(x)\overline{)P(x)}$

Example: $\dfrac{x^3+4x^2+3}{x^2+2x+1}$

<u>Long Division</u>

Divide:

$$\begin{array}{r} x+2 \\ x^2+2x+1\overline{)\,x^3+4x^2+0x+3} \\ -(x^3+2x^2+x) \\ \hline 2x^2-\ \ x+3 \\ -(2x^2+4x+2) \\ \hline -5x+1 \end{array}$$

Result:

$$\frac{x^3+4x^2+3}{x^2+2x+1}=x+2+\frac{-5x+1}{x^2+2x+1}$$

2. If you have a **proper rational function** (the degree of the numerator is less than the degree of the denominator) then you are ready to proceed to the step of partial fractions.

Partial Fractions

Step 1: Factor the denominator completely into a product of linear and/or irreducible quadratic factors with real coefficients.

Examples:

a) $\dfrac{4x-1}{2x^2-x-3} = \dfrac{4x-1}{(x+1)(2x-3)}$

b) $\dfrac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \dfrac{2x^3-4x-8}{x(x-1)(x^2+4)}$

Step 2: Rewrite the original fraction into a series of partial fractions using the following forms:

CASE 1: The denominator *Q(x)* is a product of distinct linear factors.

For each linear factor use one corresponding fraction of the form $\dfrac{A}{ax+b}$ where A is a constant to be determined.

Example: $\dfrac{4x-1}{(x+1)(2x-3)} = \dfrac{A}{x+1}+\dfrac{B}{2x-3}$

CASE 2: *Q(x)* is a product of linear factors, some of which are repeated.

For a linear factor that is repeated n times write n corresponding partial fractions of the form:

$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \dots + \dfrac{A_n}{(ax+b)^n}$ where $A_1, A_2, \dots A_n$ are constants

Example: $\dfrac{3x^2-3}{(x-1)^3} = \dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}$

CASE 3: *Q(x)* contains irreducible quadratic factors, none of which is repeated.

For each quadratic factor use one corresponding fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$ where *A, B, C, and D* are constants to be determined.

CASE 4: *Q(x)* contains a repeated irreducible quadratic factor.

For a quadratic factor that is repeated n times write n corresponding partial fractions of the form:

$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$ where $A_1, A_2, \dots A_n and B_1, B_2, \dots B_n$ are constants

<u>Step 3</u>: Determine the constants *A, B, C, D, etc.* using one of the following 2 methods.

<u>METHOD #1</u>: Solve by Equating Corresponding Coefficients

Example: $$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}$$

1. Clear all fractions by multiplying both sides by the least common denominator (LCD) $(x^2+2)(x^2+2)$:

$$8x^3+13x=(Ax+B)(x^2+2)+(Cx+D)$$

2. Remove parentheses and collect like terms on the right side of equation:

$$8x^3+13x=Ax^3+2Ax+Bx^2+2B+Cx+D$$

$$8x^3+13x=Ax^3+Bx^2+(2A+C)x+2B+D$$

3. Set corresponding coefficients equal and solve for constants *A, B, C, and D*:

$8 = A$	Coefficients of x^3
$0 = B$	Coefficients of x^2
$13 = 2A+C$	Coefficients of x
$0 = 2B+D$	Constant terms

Solving the above yields: A = 8, B = 0, C = -3, D = 0

Therefore: $$\frac{8x^3+13x}{(x^2+2)^2} = \frac{8x}{(x^2+2)} + \frac{-3x}{(x^2+2)^2}$$

Method #2: Solve by Substitution:

$$\text{Example:}\quad \frac{x-8}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)}$$

1. Clear all fractions by multiplying both sides by the LCD $(x+2)(x-3)$:

$$x-8 = A(x-3) + B(x+2)$$

2. Substitute in the values of x that make a factor on the right side of the above equation equal to 0 and solve the resulting equations:

$x = 3$

$$\begin{aligned} 3-8 &= A(3-3) + B(3+2) \\ -5 &= 0 + 5B \\ -1 &= B \end{aligned}$$

$x = -2$

$$\begin{aligned} -2-8 &= A(-2-3) + B(-2+2) \\ -10 &= -5A + 0 \\ 2 &= A \end{aligned}$$

$$\text{Therefore:}\quad \frac{x-8}{(x+2)(x-3)} = \frac{2}{(x+2)} + \frac{-1}{(x-3)}$$

Additional example using method #2:

$$\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+4)}$$

1. Clear the equation of fractions by multiplying by the LCD $(x+2)(x-3)$

   $$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

2. Substitute in values of x that make a factor on the right side equal zero.

   $x=1$

   $$\begin{aligned} 2-4-8 &= 0 + B(1)(5) + 0 \\ -10 &= 5B \\ -2 &= B \end{aligned}$$

   $x=0$

   $$\begin{aligned} 0-0-8 &= A(-1)(4) + 0 + 0 \\ -8 &= -4A \\ 2 &= A \end{aligned}$$

3. Pick <u>any other convenient values</u> for x and substitute into the equation to find C and D. Set $A=2$ and $B=-2$

   $$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

   Let $x=-1$:

   $$\begin{aligned} -2+4-8 &= 2(-2)(5) + (-2)(-1)(5) + (-C+D)(-1)(-2) \\ -6 &= -10 + 2(-C+D) \\ 2 &= -C+D \end{aligned}$$

   Let $x=2$:

   $$\begin{aligned} 16-8-8 &= 2(1)(8) + (-2)(2)(8) + (2C+D)(2)(1) \\ 0 &= -16 + 2(2C+D) \\ 8 &= 2C+D \end{aligned}$$

   Solve simultaneously:

   $$\begin{aligned} 2 &= -C+D \\ 8 &= 2C+D \end{aligned} \longrightarrow \begin{aligned} C &= 2 \\ D &= 4 \end{aligned}$$

   Therefore: $$\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{2}{x} + \frac{-2}{(x-1)} + \frac{2x+4}{(x^2+4)}$$

Using Partial Fractions Technique to Evaluate an Integral:

Example: Evaluate the following indefinite integral.

$$\int \frac{x^3+4x^2+3}{x^2+2x+1}\,dx$$

Solution:

Change the improper fraction to a polynomial plus a proper fraction using long division.

$$\int \frac{x^3+4x^2+3}{x^2+2x+1}\,dx \quad = \int \left(x+2+\frac{-5x+1}{x^2+2x+1} \right) dx$$

Write the proper fraction as partial fractions and solve for A & B. Either method would work. This example demonstrates method #1.

$$\frac{-5x+1}{x^2+2x+1} \;=\; \frac{-5x+1}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$$

$$-5x+1=A(x+1)+B$$

$$-5x+1=Ax+A+B$$

$-5=A$ ⟵ Coefficients of x.

$1=A+B$ ⟵ Constant terms.

Therefore $B=6$.

Write the integral with partial fractions and then evaluate.

$$\int \frac{x^3+4x^2+3}{x^2+2x+1}\,dx \quad = \int \left(x+2+\frac{-5}{x+1}+\frac{6}{(x+1)^2} \right) dx$$

$$=\frac{x^2}{2}+2x-5\ln|x+1|-\frac{6}{x+1}+C$$

Practice Problems

Write the following as partial fractions:

1. $\dfrac{x}{x^2-4x-5}$

2. $\dfrac{2x+1}{x^2+2x+1}$

3. $\dfrac{1}{x^3+x^2+x}$

4. $\dfrac{x+3}{x^3-4x}$

Evaluate the following integrals:

5. $\displaystyle\int \frac{dx}{x^2-4}$

6. $\displaystyle\int \frac{x+1}{x^3+x^2-6x}\,dx$

7. $\displaystyle\int \frac{2x^3}{(x^2+1)^2}\,dx$

8. $\displaystyle\int \frac{x^4-x^3-x-1}{x^3-x^2}\,dx$

Solutions to practice problems:

1. $\dfrac{x}{x^2-4x-5}=\dfrac{x}{(x-5)(x+1)}=\dfrac{A}{x-5}+\dfrac{B}{x+1}=$ $\boxed{\dfrac{5}{6(x-5)}+\dfrac{1}{6(x+1)}}$

2. $\dfrac{2x+1}{x^2+2x+1}=\dfrac{2x+1}{(x+1)^2}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}=$ $\boxed{\dfrac{2}{x+1}-\dfrac{1}{(x-1)^2}}$

3. $\dfrac{1}{x^3+x^2+x}=\dfrac{1}{x(x^2+x+1)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+x+1}=$ $\boxed{\dfrac{1}{x}-\dfrac{x+1}{x^2+x+1}}$

4. $\dfrac{x+3}{x^3-4x}=\dfrac{x+3}{x(x+2)(x-2)}=\dfrac{A}{x}+\dfrac{B}{x+2}+\dfrac{C}{x-2}=$ $\boxed{\dfrac{-3}{4x}+\dfrac{1}{8(x+2)}+\dfrac{5}{8(x-2)}}$

5.

$\displaystyle\int\frac{dx}{x^2-4}=\quad\frac{1}{4}\int\left(\frac{1}{x-2}-\frac{1}{x+2}\right)dx=\quad\frac{1}{4}\left[\ln|x-2|-\ln|x+2|\right]+C$

$\boxed{\dfrac{1}{4}\ln\left|\dfrac{x-2}{x+2}\right|+C}$

6.

$\displaystyle\int\frac{x+1}{x^3+x^2-6x}\,dx=\quad\int\left(\frac{-1}{6x}+\frac{3}{10(x-2)}-\frac{2}{15(x+3)}\right)dx=$

$\boxed{\dfrac{-1}{6}\ln|x|+\dfrac{3}{10}\ln|x-2|-\dfrac{2}{15}\ln|x+3|+C}$

7.

$\displaystyle\int\frac{2x^3}{(x^2+1)^2}\,dx=\quad\int\left(\frac{2x}{x^2+1}+\frac{-2x}{(x^2+1)^2}\right)dx=$

$\boxed{\ln(x^2+1)+\dfrac{1}{x^2+1}+C}$

8. $\displaystyle\int\frac{x^4-x^3-x-1}{x^3-x^2}\,dx=\quad\int\left(x-\frac{x+1}{x^2(x-1)}\right)dx=\int\left(x+\frac{2}{x}+\frac{1}{x^2}-\frac{2}{x-1}\right)dx=$

$\boxed{\dfrac{x^2}{2}+2\ln|x|-\dfrac{1}{x}-2\ln|x-1|+C}$