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Convergence Tests for Infinite Series

NAME	COMMENTS	STATEMENT
Geometric series	$\sum ar^k = \frac{a}{1-r}$, if $-1 < r < 1$	Geometric series converges if $-1 < r < 1$ and diverges otherwise
Divergence test (nth Term test)	If $\lim_{k \to \infty} a_k \neq 0$, then $\sum a_k$ diverges.	If $\lim_{k \to \infty} a_k = 0$, $\sum a_k$ may or may not converge.
p – series	If p is a real constant, the series $\sum \frac{1}{a^p} = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} + \ldots$ converges if $p > 1$ and diverges if $0 < p \leq 1$.	
Integral test	$\sum a_k$ has positive terms, let f(x) be a function that results when k is replace by x in the formula for u_k. If is decreasing and continuous for $x \geq 1$, then $\sum a_k$ and $\int_1^{\infty} f(x)\,dx$ both eonverge or both diverge.	Use this test when f(x) is easy to integrate. This test only applies to series with positive terms.
Comparison test (Direct)	If $\sum a_k$ and $\sum b_k$ are series with positive terms such that each term in $\sum a_k$ is less than its corresponding term in $\sum b_k$, then (a) if the "bigger series" $\sum b_k$ converges, then the "smaller series" $\sum a_k$ converges. (b) if the "smaller series" $\sum a_k$ diverges, then the "bigger series" $\sum b_k$ diverges.	Use this test as a last resort. Other test are often easier to apply. This test only applies to series with positive terms.
Limit Comparison test	If $\sum a_k$ and $\sum b_k$ are series with positive terms such that $\lim_{k \to \infty} \frac{a_k}{b_k} = L$ if $L > 0$, then then both series converge or both diverge. if $L = 0$, and $\sum b_k$ converges, then $\sum a_k$ converges. if $L = +\infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.	This is easier to apply than the comparison test, but still requires some skill in choosing the series $\sum b_k$ for comparison.
Ratio test	If $\sum a_k$ is a series with positive terms such that $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = L$, then if $L < 1$, the series converges if $L > 1$ or $L = +\infty$, the series diverges if $L = 1$, another test must be used.	Try this test when a_k involves factorials or kth powers.
Root test	If $\sum a_k$ is a series with positive terms such that $\lim_{k \to \infty} \sqrt[k]{a_k} = \lim_{k \to \infty} (a_k)^{1/k} = L$, then if $L < 1$, the series converges if $L > 1$ or $L = +\infty$, the series diverges if $L = 1$, another test must be used.	Try this test when a_k involves kth powers.
Alternating Series test (Leibniz's Theorem)	The series $a_1 - a_2 + a_3 - a_4 + \ldots$ and $-a_1 + a_2 - a_3 + a_4 - \ldots$ converge if (1) $a_1 \geq a_2 \geq a_3 \geq \ldots$ and (2) $\lim_{k \to \infty} a_k = 0$ The series diverges if $\lim_{k \to \infty} a_k \neq 0$	Alternating Series Estimation Theorem: If the alternating series $\sum (-1)^{k+1} a_k$ converges, then the truncation error for the n^{th} partial sum is less than a_{n+1}, i.e. if an alternating series converges, then the error in estimating the sum using n terms is less than the $n+1^{st}$ term.
Absolute Convergence and Conditional Convergence	If $\sum a_k$ is a series with nonzero terms that converges, then: if $\sum \lvert a_k \rvert$ converges, then $\sum a_k$ converges absolutely. if $\sum \lvert a_k \rvert$ diverges, then $\sum a_k$ converges conditionally. Otherwise, $\sum a_k$ diverges.	Note that if a series converges absolutely, then it converges, i.e. if $\sum \lvert a_k \rvert$ converges, then $\sum a_k$ converges.

Strategies for Testing Series

You have now studied ten tests for determining the convergence or divergence of an infinite series. Skill in choosing and applying the various tests will come only with practice. Below is a set of guidelines for choosing an appropriate test.

Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test.

Example **Applying the Strategies for Testing Series**

Determine the convergence or divergence of each series.

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ **e.** $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ **f.** $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Solution

a. For this series, the limit of the nth term is not 0 $\left(a_n \to \frac{1}{3} \text{ as } n \to \infty\right)$. So, by the nth-Term Test, the series diverges.

b. This series is geometric. Moreover, because the common ratio of the terms is less than 1 in absolute value $(r = \pi/6)$, you can conclude that the series converges.

c. Because the function $f(x) = xe^{-x^2}$ is easily integrated, you can use the Integral Test to conclude that the series converges.

d. The nth term of this series can be compared to the nth term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.

e. This is an alternating series whose nth term approaches 0. Because $a_{n+1} \leq a_n$, you can use the Alternating Series Test to conclude that the series converges.

f. The nth term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

g. The nth term of this series involves a variable that is raised to the nth power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges.

Source: *Calculus with Analytic Geometry*, 7$^{\text{th}}$ Ediition, by Ron Larson, Robert P. Hostetler, and Bruce H. Edwards, pg. 601

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Strategies for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. The value of p should be chosen as in example 2 by keeping only the highest powers of n in the numerator and denominator. The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence. □

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

EXAMPLE 3 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^\infty xe^{-x^2}\, dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works. □

EXAMPLE 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test. □

EXAMPLE 5 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test. □

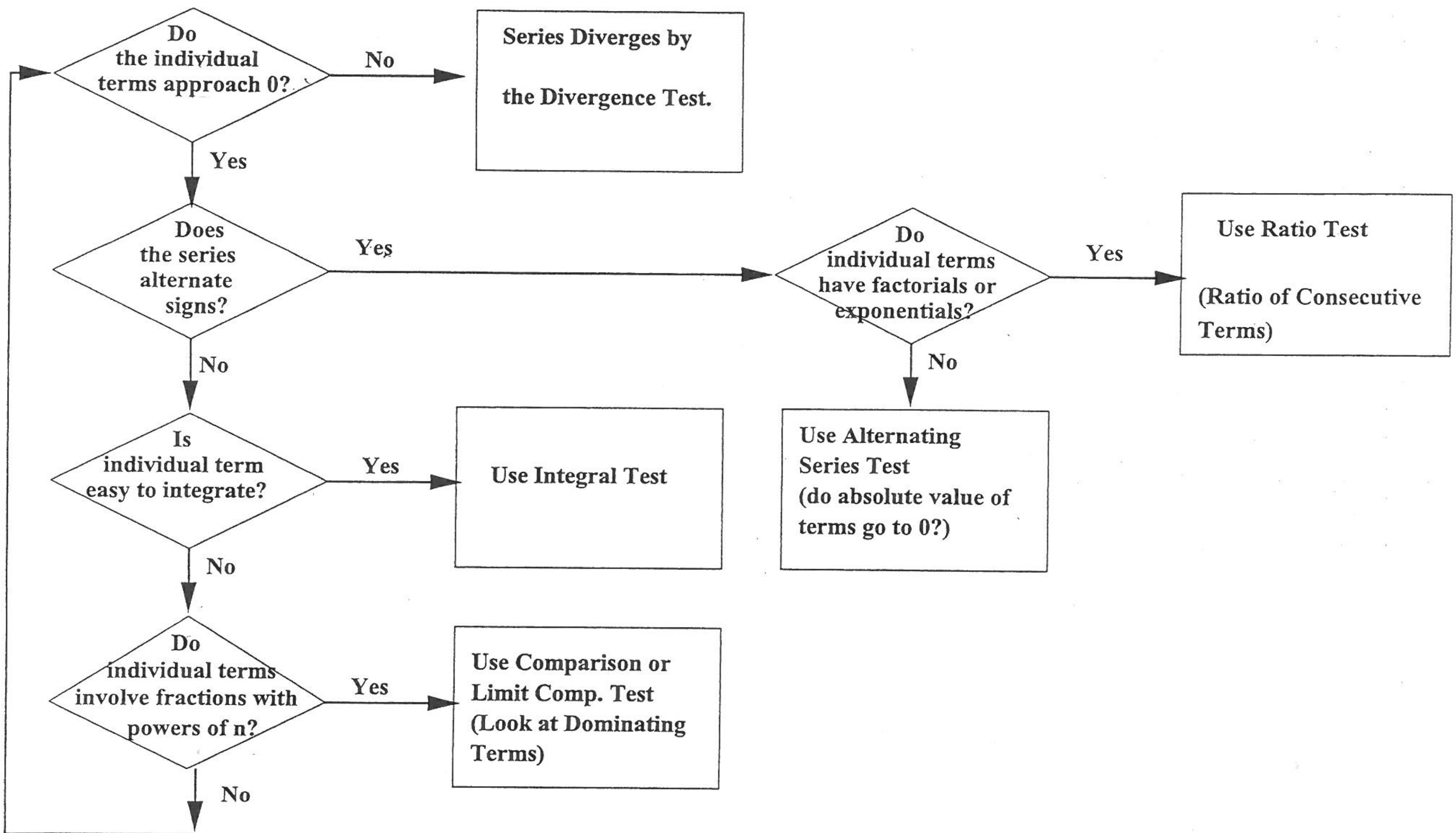
EXAMPLE 6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\Sigma\, 1/3^n$, we use the Comparison Test. □

Source: James Stewart's *Calculus: Early Transcendentals Edition 6E*, pg. 771-2.

Choosing a Convergence Test for Infinite Series

Courtesy David J. Manuel



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