First-Order Differential Equations

A

TYPE	FORMAT	METHOD
Separable	$g(y)dy = h(x)dx$	1. Integrate both sides and the equation becomes $G(y) = H(x) + C$ 2. Solve the equation explicitly for y if possible.

General Example:

Solve $3yy' = \frac{1+3x^2}{y-2}$; $y(0) = 1$ (separate variables)

$\therefore \frac{dy}{dx} = \frac{1+3x^2}{3y(y-2)} = \frac{1+3x^2}{3y^2-6y}$ $\therefore$ $(3y^2 - 6y)dy = (1 + 3x^2)dx$ (now integrate both sides)

Thus, $y^3 - 3y^2 = x + x^3 + C$ (now find the value of C using $y(0) = 1$, $[i.e., \ x = 0, y = 1]$)

$1^3 - 3 \cdot 1^2 = 0 + 0^3 + C$ yields $C = -2$

Hence, $y^3 - 3y^2 = x^3 + x + 2$, which cannot be solved explicitly for y. ■

Now you try one.

Solve $\frac{dy}{dx} = (1 + y^2)\tan x$; $y(0) = \sqrt{3}$

B

TYPE	FORMAT	METHOD
Linear	$\frac{dy}{dx} + p(x)y = q(x)$ "Linear Form"	1. Find the Integrating Factor: $\mu(x) = e^{\int p(x)dx}$ 2. The equation becomes $\mu(x) \cdot y = \int \mu(x) \cdot q(x)dx$ 3. Perform the integration and solve for y by diving both sides of the equation by $\mu(x)$.

General Example:

Solve $\frac{dy}{dx} = \frac{y}{x} + 2x + 1$ with $y(0) = e$. First put into "linear form" $\therefore \frac{dy}{dx} - \frac{1}{x} \cdot y = 2x + 1$.

Find the integration factor, $\mu(x) = e^{\int p(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-lnx} = x^{-1}$

Therefore, $x^{-1} \cdot \frac{dy}{dx} - (x^{-1}) \cdot \frac{1}{x} y = x^{-1} \cdot (2x + 1)$ or $x^{-1}\frac{dy}{dx} - x^{-2}y = 2 + x^{-1}$

Note that, using the product rule, $d(x^{-1}y) = x^{-1}\frac{dy}{dx} - x^{-2}y$, so that $d(x^{-1}y) = 2 + x^{-1}dx$

Integrating both sides yields $x^{-1}y = 2x + ln|x| + c$ and, solving for y, $y = 2x^2 + xln|x| + Cx$

Using initial conditions to solve for C: $0 = 2e^2 + eln|e| + eC = 2e^2 + e + eC = 0$, so $C = -2e - 1$

Hence, $y = 2x^2 + xln|x| - 2e - 1$. ■

Now you try one.

Solve $(x + y + 1)dx - dy = 0;$

C

TYPE	FORMAT	METHOD
Exact	$M(x,y)dx + N(x,y)dy = 0$ where $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$	1. We are looking for a solution of the form $F(x,y) = C$ where $\frac{\partial F}{\partial y} = M(x,y)$ *and* $\frac{\partial F}{\partial x} = N(x,y)$, therefore $F(x,y) = \int M(x,y)dx + g(y)$ 2. $g(y)$ is the constant of integration and can be found by differentiating both sides with respect to y, solving for $g'(y)$and integrating to get $g(y) = \int \left[N(x,y) - \frac{\partial}{\partial y}(\int M(x,y)dx)\right] dy$ (see example) 3. $F(x,y)$ can be found by substituting the equation found for $g(y)$back into the equation for F(x,y) found in step 1 and performing the indicated operations. The final solution is $F(x,y) = C$ 4. If given initial conditions, C can be made explicit. **Note:** The solution can also be found by starting with the equation: $F(x,y) = \int N(x,y)dy + h(x)$ and then solving for h(x).

General Example: **Solve** $(2xy - \sec^2 x) + (x^2 + 2y)dy = 0$ [given in proper form]

∴ take partial derivatives of each side: $\frac{\partial(2xy - \sec^2 x)}{\partial y} = 2x$ and $\frac{\partial(x^2+2y)}{\partial x} = 2x$.

Since the two partial derivatives are equal, the differential equation is "exact". Hence,

1. $F(x,y) = \int(2xy - \sec^2 x)\, dx = x^2y - \tan x + g(y) = C$

2. $\frac{\partial\left(x^2-\tan x+g(y)\right)}{dy} = \cancel{x^2} + g'(y) = \cancel{x^2} + 2y$

$$\therefore \quad g(y) = \int 2y\, dy = y^2$$

3. Hence, $F(x, y) = x^2 y - \tan x + y^2 = C$ (implicit solution – cannot be solved for y)

Given initial condition $y(0) = 5,\ 0^2 \cdot y - \tan 0 + 5^2 = C$ yields $C = 25$.

$\therefore\ x^2 y - \tan x + y^2 = 25.$ ■

Now you try one.

Solve $[e^x(y - x)]dt + (1 + e^x)dy = 0;\quad y(0) = 2$

D

TYPE	FORMAT	METHOD
Homogeneous Substitution	$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$	1. Let $v = \frac{y}{x}$ and rewrite the equation as $v + x\frac{dy}{dx} = g(v)$ 2. This equations is **separable**. Use that method to solve, then substitute $\frac{y}{x}$ for v in the solution.

General Example: **Solve** $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$. First show this is homogeneous.

$$\frac{dy}{dx} = \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2} = 1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = 1 + v + v^2 \text{ where } v = \frac{y}{x}$$

Since $v = \frac{y}{x}$, $y = vx$ and, taking the derivative of both sides, $\frac{dy}{dx} = v + x\frac{dv}{dx}$ (to be substituted)

$\therefore\ \frac{dy}{dx} = v + x\frac{dv}{dx} = 1 + v + v^2$ which can be separated

$\therefore\ \left[\frac{1}{1+v^2}\right] dv = \left[\frac{1}{x}\right] dx$ and integrating both sides we get $v = \tan[\ln(x) + C]$

$\therefore\ \frac{y}{x} = \tan[\ln(x) + C]$ so $y = x\tan[\ln(x) + C]$. ■

Now you try one.

Solve: Find the implicit solution for $\frac{dy}{dx} = \frac{4y-3x}{2x-y}$

Hints:
1. The integration will require partial fractions.
2. Begin by multiplying each term by $\frac{1}{x}$.

E

TYPE	FORMAT	METHOD
Bernoulli Substitution	$\frac{dy}{dx}+p(x)y=q(x)y^n$	1. Let $v=y^{1-n}$ and rewrite the equation as $\left(\frac{1}{1-n}\right)\frac{dv}{dx}+p(x)v=q(x)$ 2. This equation is **linear**. Use that method to solve, and then substitute y^{1-n} for v in the solution.

General Example: **Solve** $\frac{dy}{dx}=xy^{-2}-2y$

First put this into the "form" of a linear equation: $\frac{dy}{dx}+2y=xy^{-2}$

This is *almost* linear. The problem lies with the y^{-2}.

Begin: $v=y^{1-n}=y^{1-(-2)}=y^3 \quad \therefore \quad v=y^3$ is one substitution to be made, and

$v=y^3 \rightarrow \frac{dv}{dx}=3y^2\frac{dy}{dx} \quad \therefore \quad \frac{dy}{dx}=\frac{1}{3y^2}\frac{dv}{dx}$ is the second substitution to be made.

So, $\frac{1}{3y^2}\frac{dv}{dx}+2y=xy^{-2}$. Rearranging, we get the following linear equation to solve: $\frac{dv}{dx}+6v=3x$.

$u(x)=e^{\int 6dx}=e^{6x} \quad \therefore \quad \frac{dv}{dx}\cdot e^{6x}+6v\cdot e^{6x}=3x\cdot e^{6x}$, so, $\int\frac{d(ve^{6x})}{dx}=3\int x\,e^{6x}\,dx$ and

integrating by parts (do this now) you get $ve^{6x}=\frac{1}{2}xe^{6x}-\frac{1}{12}e^{6x}+C$

$\therefore \quad v=\frac{1}{2}x-\frac{1}{12}+Ce^{-6x} \quad \therefore \quad y^3=\frac{1}{2}x-\frac{1}{12}+Ce^{-6x} \quad \therefore \quad y=\sqrt[3]{\frac{1}{2}x-\frac{1}{12}+Ce^{-6x}}$. ■

Now you try one.

Solve: $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$

SOLUTIONS

A. $y=\tan\left[\frac{\pi}{3}-ln|\cos x|\right]$

B. $y=-(x+2)+Ce^x$

C. $y=\frac{xe^x-e^x+5}{1+e^x}$

D. Type equation here.

E. $y=\frac{2}{2cx-x^3}$