Homogeneous Second Order Differential Equations

To determine the general solution to homogeneous second order differential equation:

$$y''+p(x)y'+q(x)y=0$$

Find two linearly independent solutions y_1 and y_2 using one of the methods below.

Note that y_1 and y_2 are **linearly independent** if there exists an x_o such that Wronskian

$$W(y_1,y_2)(x_0)=\det\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0)\end{vmatrix}=y_1(x_0)y_2'(x_0)-y_2(x_0)y_1'(x_0)\neq 0$$

The general solution is $y(x)=C_1y(x)+C_2y(x)$ where C_1 and C_2 are arbitrary constants.

METHODS FOR FINDING TWO LINEARLY INDEPENDENT SOLUTIONS		
Method	**Restrictions**	**Procedure**
Reduction of order	Given one non-trivial solution $f(x)$ to $y''+p(x)y'+q(x)y=0$	Either: 1. Set $y(x)=v(x)\cdot f(x)$ for some unknown $v(x)$ and substitute into differential equation. 2. Now we have a separable equation in v' and v''. Use the Integrating Factor Method to get v' and then integrate to get v. 3. Substitute v back into $y(x)=v(x)\cdot f(x)$ to get the second linearly independent solution. Or: $y(x)=f(x)\cdot\int\frac{e^{-\int p(x)dx}}{[f(x)]^2}dx$ where $y(x)$ is the second linearly independent solution.
Characteristic (Auxiliary) Equation $ar^2+br+c=0$	$ay''+by'+cy=0$ where a, b and c are constants	1. Find solutions r_1 and r_2 to the characteristic (auxiliary) equation: $ar^2+br+c=0$ 2. The two linearly independent solutions are: a. If r_1 and r_2 are two real, distinct roots of characteristic equation : $y_1=e^{r_1x}$ and $y_2=e^{r_2x}$ b. If $r_1=r_2$ then $y_1=e^{r_1x}$ and $y_2=xe^{r_1x}$. c. If r_1 and r_2 are complex, conjugate solutions: $\alpha\pm\beta i$ then $y_1=e^{\alpha x}\cos\beta x$ and $y_2=e^{\alpha x}\sin\beta x$

METHODS FOR FINDING TWO LINEARLY INDEPENDENT SOLUTIONS (cont.)		
Method	**Restrictions**	**Procedure**
Variable Coefficients, (Cauchy-Euler)	$ax^2y'' + bxy' + cy = 0$ $x > 0$	1. Substitute $y = x^m$ into the differential equation. It simplifies to $am^2 + (b-a)m + c = 0$. If m is a solution to the characteristic equation then $y = x^m$ is a solution to the differential equation and a. If m_1 and m_2 are two real, distinct roots of characteristic equation then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ b. If $m_1 = m_2$ then $y_1 = x^m$ and $y_2 = x^m \ln x$. c. If m_1 and m_2 are complex, conjugate solutions $\alpha \pm \beta i$ then $y_1 = x^\alpha \cos(\beta \ln x)$ and $y2 = x^\alpha \sin(\beta \ln x)$

Example #1. Solve the differential equation: $2t^2y'' + ty' - 3y = 0$, given that $y_1(t) = t^{-1}$ is a solution.

Solution:

Let $y(t) = v(t) \cdot y_1(t) = v \cdot t^{-1}$

$$y'(t) = v'(t) \cdot y_1(t) + v(t) \cdot y_1'(t) = v' \cdot t^{-1} - v \cdot t^{-2}$$

$$y''(t) = v''(t) \cdot y_1(t) + 2v'(t) \cdot y_1'(t) + v(t) \cdot y''(t)_1 = v'' \cdot t^{-1} - 2v' \cdot t^{-2} + 2vt^{-3}$$

$$2t^2y'' + ty' - 3y = 0 \Rightarrow$$

$$2t^2\left(v'' \cdot t^{-1} - 2v' \cdot t^{-2} + 2vt^{-3}\right) + t\left(v' \cdot t^{-1} - v \cdot t^{-2}\right) - 3\left(v \cdot t^{-1}\right) = 0$$

$$2tv'' - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} = 0$$

$$2tv'' - 3v' = 0$$

Let $v' = u$ so $v'' = u'$ then

$$2tv'' - 3v' = 0 \Rightarrow \quad 2tu' - 3u = 0$$

$u' - \frac{3}{2t}u = 0$ (First order linear equation)

$u = t^{3/2} \Rightarrow v = \frac{2}{5}t^{5/2}$, at this point we can ignore the constant coefficients so take $v = t^{5/2}$

Substitute v back into $y(t) = v(t) \cdot y_1(t)$ to get the second linearly independent solution.

$$y_2 = v \cdot y_1 = t^{5/2} \cdot t^{-1} = t^{3/2}$$

The general solution is:

$$\boxed{\begin{aligned} y &= C_1y_1 + C_2y_2 \\ y &= C_1t^{-1} + C_2t^{3/2} \end{aligned}}$$

Example #2. Solve the differential equation: $y'' - 2y' + y = 0$

Solution:

Characteristic equation:

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0$$
$$r = 1, r = 1 \quad \text{(Repeated roots)}$$
$$\Rightarrow y_1 = C_1 e^x \text{ and } y_2 = C_2 x e^x$$

So the general solution is: $\boxed{y = C_1 e^x + C_2 x e^x}$

Example #3. Solve the differential equation: $t^2 y''(t) - 4ty'(t) + 4y(t) = 0$, given that $y(1) = -2$, $y'(1) = -11$

Solution: The substitution: $y = t^m$ yields to the characteristic equation:

$$m^2 + (-4-1)m + 4 = 0$$
$$m^2 - 5m + 4 = 0$$
$$(m-4)(m-1) = 0$$
$$m = 4 \text{ or } m = 1 \text{ two distinct, real solutions}$$

So the solutions are: t^4 and t. The general solution is

$$\boxed{y = C_1 t^4 + C_2 t}$$

Use $y(1) = -2$, $y'(1) = -11$ to find the solution to the initial value problem:

$y(1) = -2 \quad \Rightarrow C_1 + C_2 = -2$

$y'(1) = -11 \Rightarrow 4C_1 + C_2 = -11$

Solving the system of linear equations gives us $C_1 = -3$ and $C_2 = 1$

So the solution to the Initial Value Problem is $\boxed{y = t - 3t^4}$

You try it:

1. Given that $y_1(x) = e^{2x/3}$ is a solution of the following differential equation $9y'' - 12y' + 4y = 0$. Use the reduction of order to find a second solution. (Hint: $v'' = 0$ implies $v' = 1$)

Find the general solution of the given second-order differential equations:

2. $3y'' + 2y' + y = 0$
3. $x^2 y'' + 5xy' + 4y = 0$

Solutions:

#1: $y_2 = xe^{2x/3}$

#2: $y = e^{-x/3}\left[C_1 \cos\left(\frac{\sqrt{2}}{3}x\right) + C_2 \sin\left(\frac{\sqrt{2}}{3}x\right)\right]$

#3: $y = C_1 x^{-2} + C_2 x^{-2} \ln x$