Laplace Transforms and Non-Standard Functions

Solving Differential Equations with Laplace Transforms:

1. Take the Laplace transform of both sides of the equation.
2. Using the initial conditions, solve the equation for *Y(s)*.
3. Take the inverse Laplace of both sides of the equation to find *y(t)*.

Inverse Laplace Transforms of Rational Functions

Determine which Laplace Transform you will use by examining the denominator. Then using algebraic procedures make the numerator fit the form. You may need to do one or more of the following, before the denominator will match one of the forms from the table.

1. Divide, if the degree of a polynomial in the numerator is greater than or equal to the degree of a polynomial in the denominator.
2. If the denominator can be factored -rewrite as 2 or more rational expressions using method of partial fractions.
3. If the denominator cannot be factored – complete the square.

Example #1. Use the Laplace transform to solve the given initial-value problem:

$$y'' + 4y = t, \quad y(0) = 0, y'(0) = 0$$

Solution:

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{t\}$$

$$\left[s^2Y(s) - sy(0) - y'(s)\right] + 4Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) + 4Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2} \cdot \frac{1}{s^2+4}$$ using partial fractions we get $Y(s) = \frac{A}{s^2} + \frac{B}{s^2+4}$

Solving for A and B yields to $A = \frac{1}{4}$ and $B = -\frac{1}{4}$

By partial fractions:

$$Y(s) = \frac{1}{4} \cdot \frac{1}{s^2} - \frac{1}{4} \cdot \frac{1}{s^2+4} = \frac{1}{4} \cdot \frac{1}{s^2} - \frac{1}{8} \cdot \frac{2}{s^2+2^2}$$

Denominator is made to match one of the forms from the Laplace Transforms Table (in this case it is sin(kt))

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4} \cdot \frac{1}{s^2} - \frac{1}{4} \cdot \frac{1}{s^2+4}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

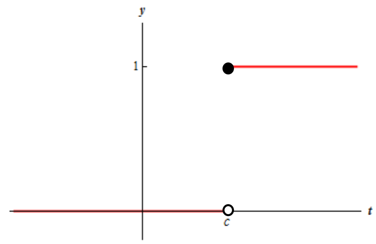
so $$\boxed{y(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)}$$

Non-Standard Functions

1. Unit Step Function or Heaviside Function denoted by either *u(t-a)* or *H(t-a)* (Turns on at a and stays on):

$$H(t-a) = u(t-a) = \begin{cases} 0 & \text{if } t-a<0 \text{ or } t<a \\ 1 & \text{if } t-a \geq 0 \text{ or } t \geq a \end{cases}$$

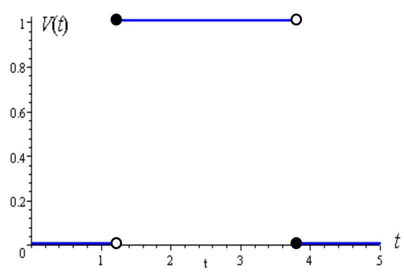
Sample graph



2. Unit Pulse Function (Turns on at a and off at b):

$$P_{a,b}(t) = u(t-a) - u(t-b) = \begin{cases} 0 & \text{if } -\infty < t < a \\ 1 & \text{if } a \leq t < b \\ 0 & \text{if } b \leq t < +\infty \end{cases}$$

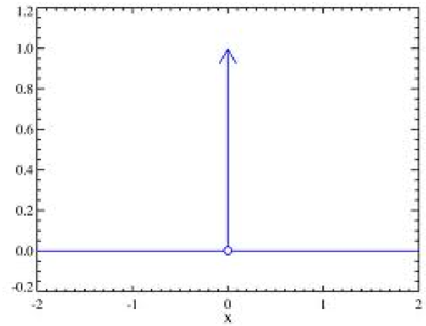
Sample graph



3. Dirac Delta Function (Unit Impulse Function):

$$\delta(t-a) = \begin{cases} 0 & \text{if } t \neq a \\ \infty & \text{if } t = a \end{cases}$$

Sample graph



Laplace transforms of unit step functions and unit pulse functions.

1. Convert unit pulse function to unit step function before taking the Laplace transform.

2. Apply the Second Translation Theorem (STT):

$$\mathcal{L}\{u(t-a)f(t)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}, \qquad (a>0)$$

Example #2. Find the Laplace transform of the following function:

$$f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ 2-t, & 1 \le t < 4 \\ 5t, & 4 \le t < \infty \end{cases}$$

Solution:
First we will rewrite *f(t)* in terms of Heaviside function:

$$f(t) = [u(t-0)-u(t-1)]\cdot t^2 + [u(t-1)-u(t-4)]\cdot(2-t) + [u(t-4)]\cdot 5t$$

And group the terms, based on the Heaviside function:

$$f(t) = u(t-0)\cdot t^2 + u(t-1)\cdot\left[(2-t)-t^2\right] + u(t-4)\cdot[5t-(2-t)] =$$
$$u(t-0)t^2 + u(t-1)\left(2-t-t^2\right) + u(t-4)\cdot(6t-2)$$

Notice that $u(t-0)=1$ for all $t \ge 0$ so *f(t)* becomes:

$$f(t) = t^2 + u(t-1)\left(2-t-t^2\right) + u(t-4)\cdot(6t-2)$$

Now we are ready to find the Laplace transform of *f(t):*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^2 + u(t-1)\left(2-t-t^2\right) + u(t-4)\cdot(6t-2)\right\} =$$

$$= \mathcal{L}\left\{t^2\right\} + \mathcal{L}\left\{u(t-1)\left(2-t-t^2\right)\right\} + \mathcal{L}\{u(t-4)\cdot(6t-2)\} =$$

$$= \mathcal{L}\left\{t^2\right\} + e^{-s}\,\mathcal{L}\left\{2-(t+1)-(t+1)^2\right\} + e^{-4s}\,\mathcal{L}\{6(t+4)-2\} =$$

$= \mathcal{L}\{t^2\} + e^{-s}\mathcal{L}\{-3t - t^2\} + e^{-4s}\mathcal{L}\{6t + 22\} =$

$= \mathcal{L}\{t^2\} - 3e^{-s}\mathcal{L}\{t\} - e^{-s}\mathcal{L}\{t^2\} + 6e^{-4s}\mathcal{L}\{t\} + 22e^{-4s}\mathcal{L}\{1\} =$

$$= \frac{2}{s^3} - \frac{3e^{-s}}{s^2} - \frac{2e^{-s}}{s^3} + \frac{6e^{-4s}}{s^2} + \frac{22e^{-4s}}{s} = \boxed{\frac{2 - e^{-s}(3s+2) + 2se^{-4s}(3+11s)}{s^3}}$$

<u>Inverse Laplace transform of the unit step functions.</u>

Apply the Second Translation Theorem (STT):

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a), \qquad (a > 0)}$$

Example #3. Find the inverse transform of $F(s) = \frac{1 - e^{-2s}}{s^2}$

Solution:

$$F(s) = \frac{1-e^{-2s}}{s^2} = \frac{1}{s^2} - e^{-2s}\cdot\frac{1}{s^2} \quad \text{so}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - e^{-2s}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t - u(t-2)\cdot(t-2)$$

which can also be written as:

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & 2 \le t < \infty \end{cases}$$

You try it:

1. Use the Laplace transform to solve the given initial-value problem: $y'' + 2y' - 15y = 8e^{-t}, \quad y(0) = 2, y'(0) = -12$
2. Find the Laplace transform of $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ t+6, & t \ge 1 \end{cases}$
3. Find the inverse transform of $F(s) = \frac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

Answers:

1. $y = \frac{5}{2}e^{-5t} - \frac{1}{2}e^{-t}$
2. $F(s) = \frac{2}{s^3} - \frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2} + \frac{6e^{-s}}{s}$
3. $f(t) = 2u(t-2)e^{t-2}\cos(t-2)$