Disjunctive Normal Form or Sum-of-Products Form

Interpret a Boolean system as a circuit. Given that you know the input into an electrical/logical network and you know what final effect that you desire, how can you find an electrical network that will produce that desired output?

This input/output system is sometimes referred to as a “black box.” A simplified example, incorporating only three switches and represented by simple propositions, looks like this:

p	q	r	?
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Notice that we clearly know all varieties of input. Our desired output is specified. We desire to know a logical expression that will produce the given output. To do this we circle each of the output labeled “true”. Considering the input (T-T-T) of the topmost circled T, we create the following **normal form**: $p \wedge q \wedge r$

Here the input is T-F-F, therefore the normal form is $p \wedge \sim q \wedge \sim r$

Here the input is F-F-T, so the normal form is $\sim p \wedge \sim q \wedge r$

There are only three true outputs; therefore there will be only three normal forms.

We join these with the disjunctive “or” resulting in the “***disjunctive*** **normal form**”:

$$(p \wedge q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r).$$

If you form the truth-table for the above expression, you will discover that it has the same truth value as given as the output found in the last column of our table.

A possible short-cut:

Consider the following “input-output” table:

p	q	?
T	T	T
T	F	F
F	T	T
F	F	T

Circling the outputted T’s and following the process given above, the disjunctive normal form will be $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$. (*Verify*!). Notice, though, that in this case there are more outputted *trues* than *falses*. The expression can be shortened, and thereby simplified, if we circle the false outputs instead.

The normal form for this is $p \wedge \sim q$, but since this matches a *false* output, it will need to be negated. Hence the normal form here is actually $\sim(p \wedge \sim q)$.

Since there are no other normal forms, this will also be considered the **disjunctive normal form**.

Interesting side note: The table given is that of the conditional statement, $p \rightarrow q$ and we have shown that its disjunctive normal form, $\sim(p \wedge \sim q)$, is logically equivalent to it. Using De Morgan’s Law provides the following result:

$$p \rightarrow q \equiv \sim(p \wedge \sim q) \equiv \sim p \vee q,$$

which is one of the important identities, one that is able to transform the higher-order implication into the more primitive “or” statement.

Now you try some:

For each of the following logical statements, find the truth value and from that information find the logically equivalent disjunctive normal form.

a. $\sim[(\sim p \rightarrow q) \rightarrow r]$

b. $P \rightarrow (q \wedge r)$

c. $(p \wedge r) \leftrightarrow (\sim q \vee r)$

ANSWERS:

a. $(p \wedge q \wedge \sim r)$ $\mathbf{V}$ $(p \wedge \sim q \wedge \sim r)$ $\mathbf{V}$ $(\sim p \wedge q \wedge \sim r)$

b. $\sim(p \wedge q \wedge \sim r)$ $\mathbf{V}$ $\sim(p \wedge \sim q \wedge r)$ $\mathbf{V}$ $\sim(p \wedge \sim q \wedge \sim r)$ (circled false outputs)

[The following would be considered correct as well: $(p \wedge q \wedge r)$ $\mathbf{V}$ $(\sim p \wedge q \wedge r)$ $\mathbf{V}$ $(\sim p \wedge q \wedge \sim r)$ $\mathbf{V}$ $(\sim p \wedge \sim q \wedge r)$ $\mathbf{V}$ $(\sim p \wedge \sim q \wedge \sim r)$.]

c. $(p \wedge q \wedge r)$ $\mathbf{V}$ $(p \wedge q \wedge \sim r)$ $\mathbf{V}$ $(p \wedge \sim q \wedge r)$ $\mathbf{V}$ $(\sim p \wedge q \wedge \sim r)$

(Note that there is the same number of true outputs as there are false outputs, therefore true outputs are chosen.)