Evaluating Trig Functions

I. Definition and Properties of the Unit Circle

a. Definition: A **Unit Circle** is the circle with a radius of one ($r = 1$), centered at the origin $(0,0)$.

b. Equation: $x^2 + y^2 = 1$

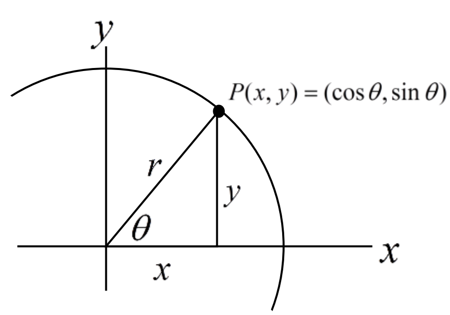
c. Arc Length
Since arc length can be found using the formula: $s = r\theta$
(where s = arc length, r = radius, θ = central angle in radians)
For the unit circle, since $r = 1$, $s = (1)\theta$
Therefore $s = \theta$

> The arc length of a sector of a unit circle equals the radian measure of angle θ.

d. Circumference: $C = 2\pi r = 2\pi(1) = 2\pi$

> The arc length (circumference) of 2π is also the radian measure of the angle corresponding to $360°$.
> 2π radians = 360 degrees
> π radians = 180 degrees

e. Relating Coordinate Values to Trig Functions
For any point $P(x, y)$ on the unit circle,
$x = \cos\theta$ and $y = \sin\theta$ where θ is any central angle with:
1) initial side = positive x axis
2) terminal side = radius through pt. P

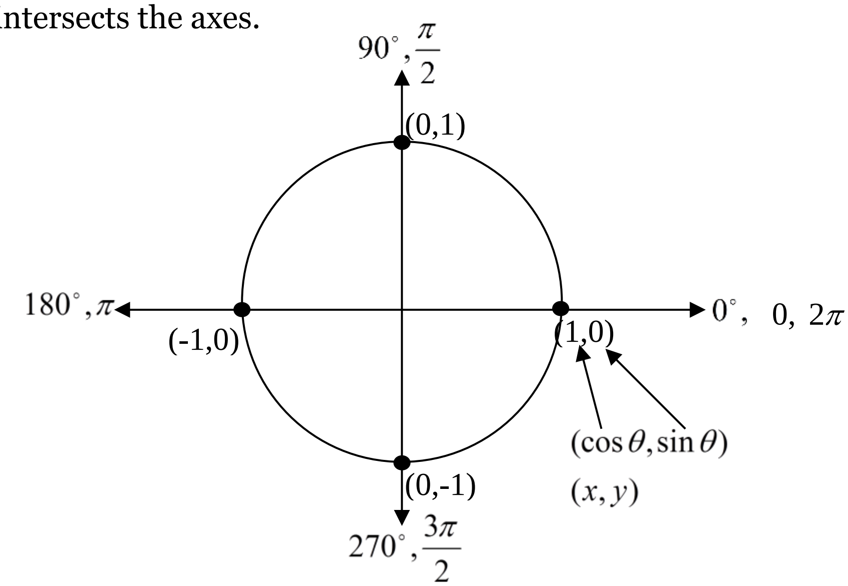


In the first quadrant this can be verified:

$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r} = \frac{y}{1} = y$$

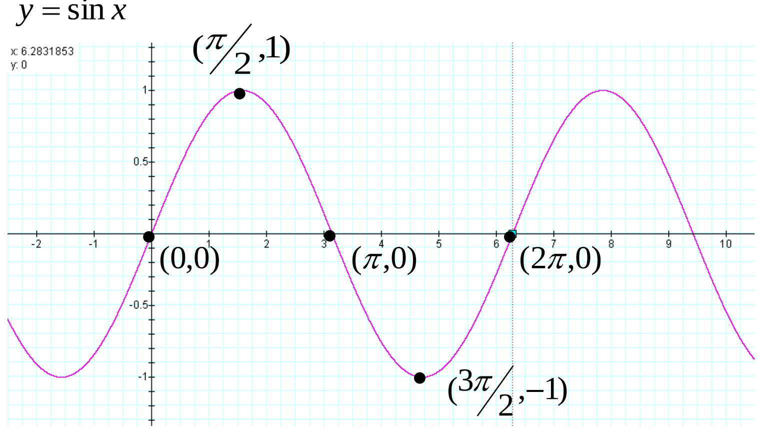
$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r} = \frac{x}{1} = x$$

f. The x and y axes can be labeled using the radian measure of the angle θ which corresponds to the points where the unit circle intersects the axes.

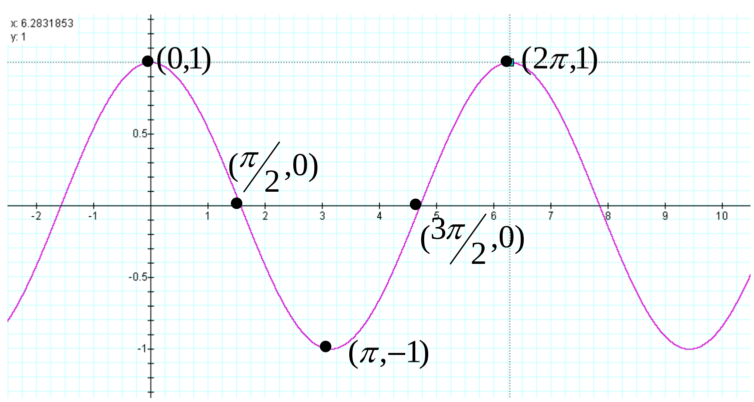


We can also see this graphically:

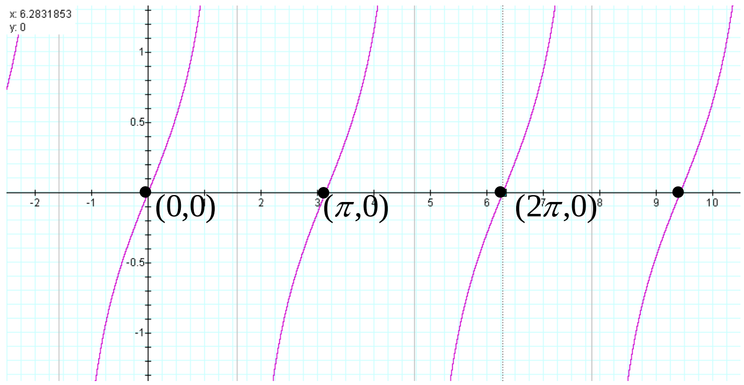
$y = \sin x$



$y = \cos x$



$y = \tan x$



Finding the sine and cosine values of quadrantal angles is now easy. For example, to find $\sin\frac{3\pi}{2}$ use the point (0,-1) which corresponds to a central angle of $\frac{3\pi}{2}$. Since $\sin\frac{3\pi}{2}$ is the y coordinate of the point, $\sin\frac{3\pi}{2} = -1$. Similarly, $\cos\frac{3\pi}{2} = 0$ (the x coordinate of the point).

g. Examples:

i. Find the arc length of a sector in a unit circle with a central angle of $120°$.

Solution:
In a unit circle, arc length = central angle measured in radians, $s = \theta$. Since π radians equals $180°$, multiply $120°$ by the conversion ratio of $\frac{\pi \quad radians}{180^\circ}$.

$120°\left(\frac{\pi}{180}\right) = \frac{2\pi}{3}$ Thus the arc length and the measure of θ are both $\frac{2\pi}{3}$ radians.

ii. Find $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$.

Solution:

$\theta=\frac{\pi}{2}$ implies the angle is a right angle ($90°$), so $P=(0,1)$. Hence, $\cos\frac{\pi}{2}=0$ (the x coordinate of P) and $\sin\frac{\pi}{2}=1$ (the y coordinate of P).

iii. Find $\sin(-\pi)$ and $\cos(-\pi)$

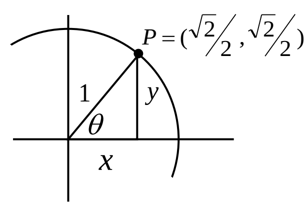
Solution:

If $\theta=-\pi$, $P=(-1,0)$. So $\sin(-\pi)=0$ (y value) and $\cos(-\pi)=-1$ (x coordinate of P).

II. More Properties of the Unit Circle

a. If $\theta=\frac{\pi}{4}$ (which is equivalent to $45°$), then for the point P on the unit circle, $x=y=\frac{\sqrt{2}}{2}$.

If $\theta=\frac{\pi}{4}$ ($45°$), then $P=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.



Explanation:

$x^2+y^2=1$ (equation of unit circle)

$x^2+x^2=1$ ($x=y$ since it is an isosceles $45^\circ-45^\circ-90^\circ$ triangle)

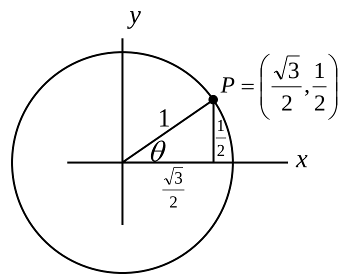
$2x^2=1$

$x=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}=y$

b. If $\theta = \frac{\pi}{6}$ (which is equivalent to $30°$), then for the point P on the unit circle, $x = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$.

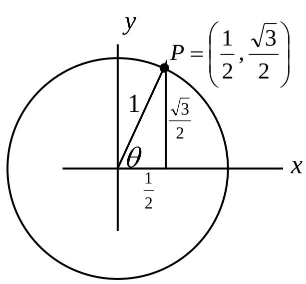
Using properties of $30^\circ - 60^\circ - 90^\circ$ triangles with hypotenuse of length 1 (since $r = 1$):

If $\theta = \frac{\pi}{6}$ $(30°)$, then $P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



c. If $\theta = \frac{\pi}{3}$ (which is equivalent to $60°$), then for the point P on the unit circle, $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

Using properties of $30^\circ - 60^\circ - 90^\circ$ triangles with hypotenuse of length 1 (since $r = 1$):

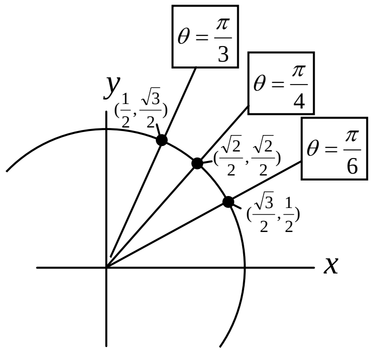


If $\theta = \frac{\pi}{3}$ $(60°)$, then $P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

NOTE:
The larger side, $\frac{\sqrt{3}}{2}$, is always opposite the larger angle, $\frac{\pi}{3}$, and the smaller side, $\frac{1}{2}$, is opposite the smaller angle, $\frac{\pi}{6}$.

d. <u>Examples:</u>
Find :

(i) $\sin\frac{\pi}{6}$ (ii) $\csc\frac{\pi}{3}$ (iii) $\tan\frac{\pi}{4}$



<u>Solution</u>:

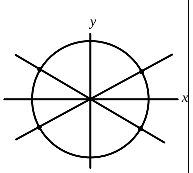
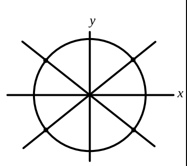
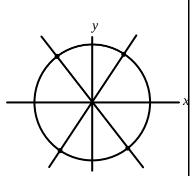
(i) $\sin\frac{\pi}{6}=\boxed{\frac{1}{2}}$ (y coordinate when $\theta=\frac{\pi}{6}$)

(ii) $\csc\frac{\pi}{3}=\frac{1}{\sin\frac{\pi}{3}}=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}=\boxed{\frac{2\sqrt{3}}{3}}$

(iii) $\tan\frac{\pi}{4}=\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\boxed{=1}$

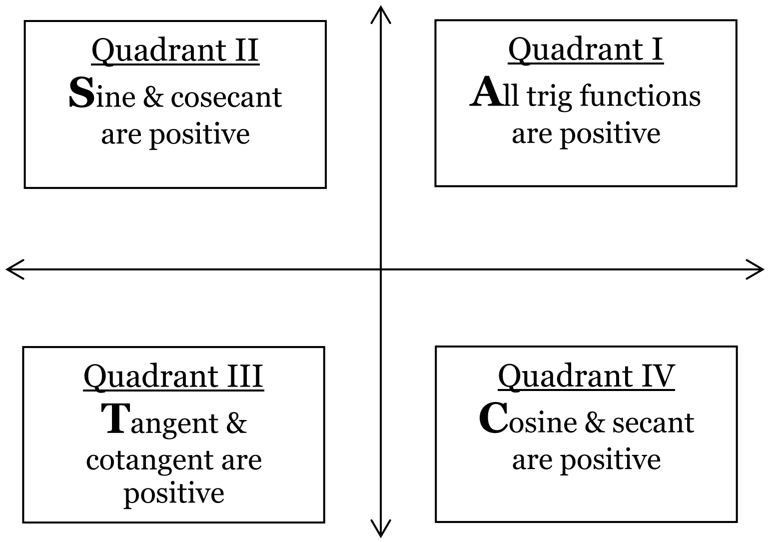
III. More on Evaluating Trig Functions Using the Unit Circle

It is important to recognize the radian measure of the standard angles related to $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ located in quadrants II, III, and IV.

Graph	Reference Angle	QI	QII	QIII	QIV	Point P
	$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\left(\pm\frac{\sqrt{3}}{2},\pm\frac{1}{2}\right)$
	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right)$
	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\left(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$

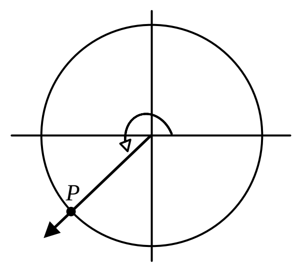
The signs of the x and y values of point P can be determined by knowing the quadrant the angle terminates in. It's as simple as remembering:

<u>**A**</u>ll <u>**S**</u>tudents <u>**T**</u>ake <u>**C**</u>alculus



<u>Examples:</u>

1. Evaluate $\sin\frac{5\pi}{4}$.



<u>Solution</u>:

$\frac{5\pi}{4}$ has a reference angle of $\frac{\pi}{4}$.

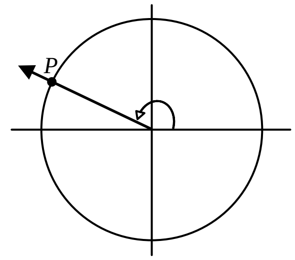
Since $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$, it is in quadrant III.

<u>NOTE</u>: Our final answer will be negative because only $\tan\theta$ and $\cot\theta$ are positive in the third quadrant.

$P = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ (since x and y are negative in QIII)

$\sin\frac{5\pi}{4} = \boxed{-\frac{\sqrt{2}}{2}}$ (the y coordinate of P)

2. Evaluate $\cos\frac{5\pi}{6}$.



Solution:
$\frac{5\pi}{6}$ has a reference angle of $\frac{\pi}{6}$.

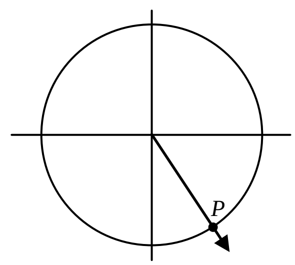
Since $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, it is in quadrant II. ←

NOTE: Our final answer will be negative because only $\sin\theta$ and $\csc\theta$ are positive in the second quadrant.

$P = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$ (since x negative in QII)

$\cos\frac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2}}$ (the x coordinate of P)

3. Evaluate $\tan\frac{11\pi}{3}$.



Solution:
$\frac{11\pi}{3}$ has a reference angle of $\frac{\pi}{3}$.

Since $\frac{11\pi}{3} = 2\pi - \frac{\pi}{3}$, it is in quadrant IV. ←

NOTE: Our final answer will be negative because only $\cos\theta$ and $\sec\theta$ are positive in the fourth quadrant.

$P = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$

$$\tan\frac{11\pi}{3} = -\frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \boxed{-\sqrt{3}}$$

IV. Other Facts Derived From the Unit Circle

1. Fundamental Identity:
For any point P on the unit circle, $P = (x, y) = (\cos\theta, \sin\theta)$.
Substituting $x = \cos\theta$ and $y = \sin\theta$ into the equation of the circle:

$$x^2 + y^2 = 1$$
$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$
$$\sin^2\theta + \cos^2\theta = 1$$

2. Other identities:
The x coordinates of points for θ and $-\theta$ are the same, so:

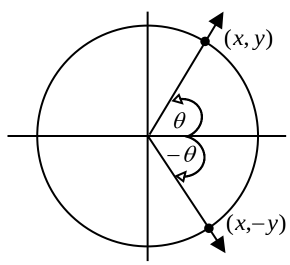
$$\cos(-\theta) = \cos(\theta)$$

Therefore $\cos(\theta)$ is an **even** function.

The y coordinates of points for θ and $-\theta$ are the opposite, so:

$$\sin(-\theta) = -\sin(\theta)$$

Therefore $\sin(\theta)$ is an **odd** function.



Practice Exercises:
Use the unit circle to answer the following problems.

1. Evaluate:
   a. $\cos\pi$
   b. $\sin\frac{3\pi}{2}$
   c. $\tan(-3\pi)$
   d. $\csc\frac{\pi}{2}$

2. Find the six trigonometric functions values at the following values of θ:
   a. $\frac{5\pi}{3}$
   b. $\frac{3\pi}{4}$
   c. $\frac{7\pi}{6}$
   d. $-\frac{11\pi}{6}$

Solutions: Unit Circle Trig

1. a) -1
   b) -1
   c) 0
   d) 1

2.

Angle	**a.** $\frac{5\pi}{3}$	**b.** $\frac{3\pi}{4}$	**c.** $\frac{7\pi}{6}$	**d.** $-\frac{11\pi}{6}$
Quadrant	QIV	QII	QIII	QI
Point P	$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\sin\theta$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\cos\theta$	$\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\theta$	$-\sqrt{3}$	-1	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$
$\csc\theta$	$-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}$	$\frac{2}{\sqrt{2}}=\sqrt{2}$	-2	2
$\sec\theta$	2	$-\frac{2}{\sqrt{2}}=-\sqrt{2}$	$-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}$	$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$
$\cot\theta$	$-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$	-1	$\sqrt{3}$	$\sqrt{3}$