Inverse Trigonometric Functions

I. **Four Facts About Functions and Their Inverse Functions:**

1. A function must be one-to-one (any horizontal line intersects it at most once) in order to have an inverse function.

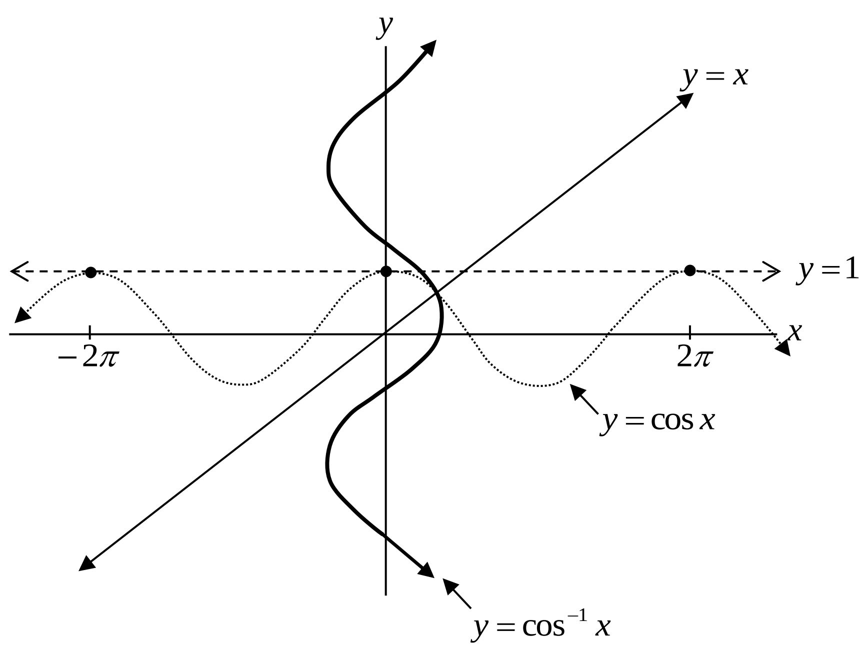
2. The graph of an inverse function is the reflection of the original function about the line $y = x$.

3. If (x, y) is a point on the graph of the original function, then (y, x) is a point on the graph of the inverse function.

4. The domain and range of a function and it's inverse are interchanged.

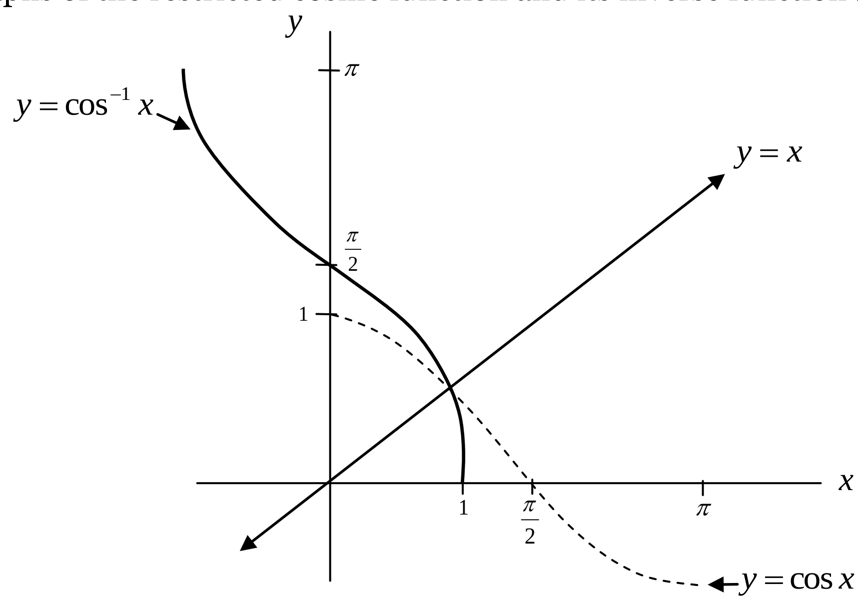
II. **Illustration of the Four Facts for the Cosine Function:**

Background: The regular cosine function for $-\infty < x < \infty$, is **not** one-to-one since some horizontal lines intersect the graph many times. (See how the horizontal line $y = 1$ intersects the portion of the cosine function graphed below in 3 places.) Therefore more than one x value is associated with a single value. The *inverse relationship* would **not** be a function as it would not pass the vertical line test.



FACT #1: A function must be one-to-one (any horizontal line intersects it at most once) in order to have an inverse function.

The **restricted cosine function**, $y = \cos x$ on the interval $0 \le x \le \pi$ **is** one-to-one and **does** have an inverse function called $\arccos x$ or $\cos^{-1} x$. See the graphs of the restricted cosine function and its inverse function below:



FACT #2: The graph of an inverse function is the reflection of the original function about the line $y = x$.

Note the symmetry of graphs of $\cos x$ and $\arccos x$ about the line $y = x$.

FACT #3: If (x, y) is a point on the graph of the original function, then (y, x) is a point on the graph of the inverse function.

$\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is a point on the graph of $y = \cos x \longrightarrow \cos\frac{\pi}{3} = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{\pi}{3}\right)$ is a point on the graph of $y = \arccos x \longrightarrow \arccos\frac{1}{2} = \frac{\pi}{3}$

In general, if $\arccos x = y$, then $x = \cos y$. ($\cos^{-1} x = y$ implies $\cos y = x$)

<u>FACT #4</u>: The domain and range of a function and it's inverse are interchanged.

	$\cos x$	$\arccos x$
Domain	$0 \le x \le \pi$ (restricted domain)	$-1 \le x \le 1$
Range	$-1 \le y \le 1$	$0 \le y \le \pi$ (restricted range)

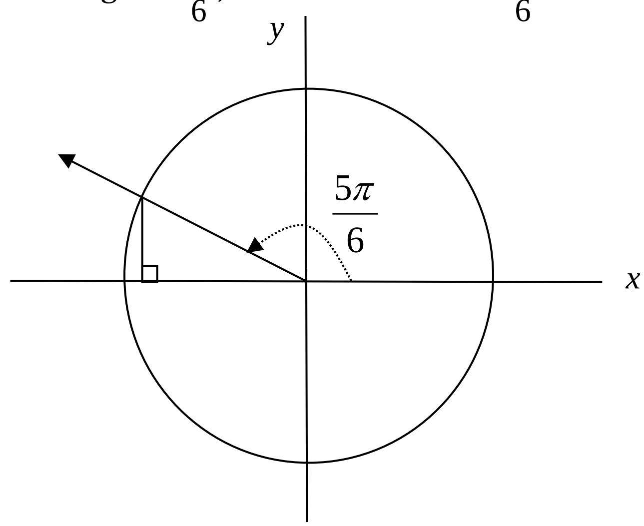
<u>Example</u>: Evaluate $\arccos\left(-\frac{\sqrt{3}}{2}\right)$.

<u>Solution</u>: The question being asked is "What angle has a cosine value of $-\frac{\sqrt{3}}{2}$?"

Usually there are an infinite number of solutions because cosine is periodic and equals this value twice each and every period. However, for the $\arccos x$ function we are looking for the answer in the **restricted range**. From the above work, we know the range of $\arccos x$ is $0 \le y \le \pi$. So the question being asked is more precisely,

"What angle between 0 and π has a cosine value of $-\frac{\sqrt{3}}{2}$?"

Since cosine is negative for angles in the 2nd quadrant and the reference angle is $\frac{\pi}{6}$, the final answer is $\frac{5\pi}{6}$.



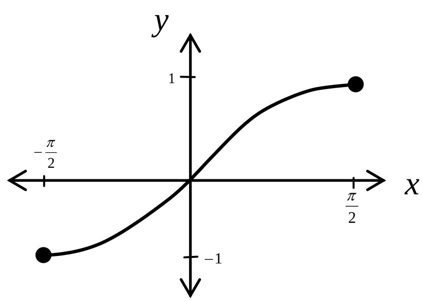
III. **Other Inverse Trigonometric Functions:**

Each trigonometric function has a restricted domain for which an inverse function is defined. The restricted domains are determined so the trig functions are one-to-one.

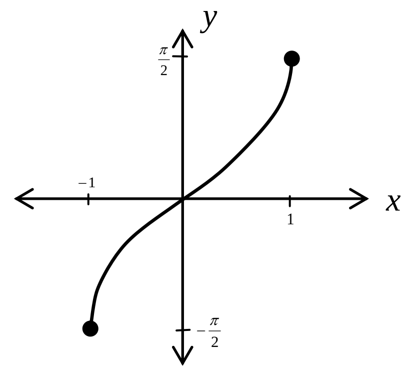
Trig function	Restricted domain	Inverse trig function	Principle value range
$y = \sin x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$y = \arcsin x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos x$	$0 \le x \le \pi$	$y = \arccos x$	$0 \le y \le \pi$
$y = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$y = \arctan x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

<u>Graphs</u>:

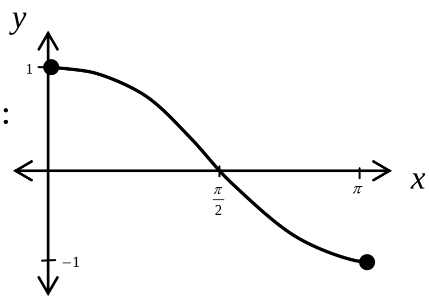
$y = \sin x$:



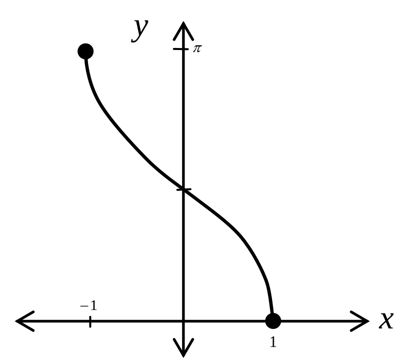
$y = \arcsin x = \sin^{-1} x$:



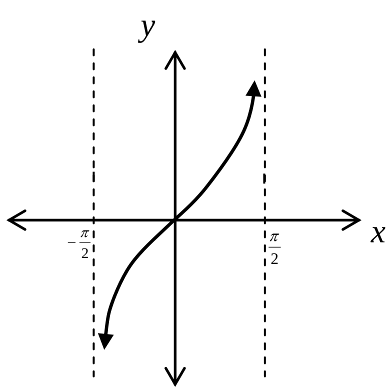
$y = \cos x$:



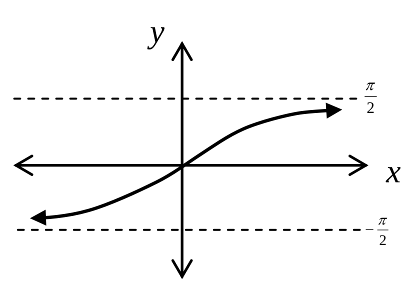
$y = \arccos x = \cos^{-1} x$:



$y = \tan x$:



$y = \arctan x = \tan^{-1} x$:



Example #1: Evaluate $y = \arcsin\left(-\frac{1}{2}\right)$.

HINT: Find the angle whose sine value equals $-\frac{1}{2}$. The answer must be in the principle range of $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Answer: $-\frac{\pi}{6}$

Example #2: Evaluate $y = \arccos\left(-\frac{1}{2}\right)$.

HINT: Find the angle whose cosine value equals $-\frac{1}{2}$. The answer must be in the principle range of $0 \le y \le \pi$.

Answer: $\frac{2\pi}{3}$

Example #3: Evaluate $y = \arctan(-1)$.

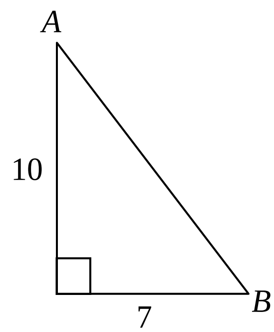
HINT: Find the angle whose tangent value equals -1. The answer must be in the principle range of $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Answer: $-\frac{\pi}{4}$

**Alternate notation for the above examples: Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$, $\cos^{-1}\left(-\frac{1}{2}\right)$, $\tan^{-1}(-1)$.

<u>Example #4</u>:

Calculator Example
If the length of two legs of a right triangle are 7 and 10, find the measure of the larger acute angle.



<u>Solution</u>:
Acute angle B is larger than angle A since the side opposite B (side $b = 10$) is larger than the side opposite A (side $a = 7$).

$$\tan B = \frac{opp}{adj} = \frac{10}{7}$$

$$B = \arctan\left(\frac{10}{7}\right) = \tan^{-1}\left(\frac{10}{7}\right)$$

$$B = 55° \quad \text{or} \quad 55\left(\frac{\pi}{180}\right) = \frac{11\pi}{36} \text{ radians}$$

<u>PROBLEMS</u>:
1. Evaluate:

a) $\arccos(0)$ b) $\arccos\left(\frac{\sqrt{2}}{2}\right)$ c) $\cos^{-1}(-1)$ d) $\cos^{-1}(1)$

e) $\sin^{-1}(0)$ f) $\arcsin(-\frac{\sqrt{3}}{2})$ g) $\arctan(\sqrt{3})$ h) $\tan^{-1}(-\sqrt{3})$

2. Find the measure of the acute angles in a right triangle with a hypotenuse of length 10 and a side of length 7.

ANSWERS:

1. a) $\frac{\pi}{2}$ b) $\frac{\pi}{4}$ c) π d) 0

e) 0 f) $-\frac{\pi}{3}$ g) $\frac{\pi}{3}$ h) $-\frac{\pi}{3}$

2. $\sin^{-1}\left(\frac{7}{10}\right) = \arcsin\left(\frac{7}{10}\right) = 44.4°$. The other angle $90° - 44.4° = 45.6°$.

Alternate solution: $\cos^{-1}\left(\frac{7}{10}\right) = \arccos\left(\frac{7}{10}\right) = 45.6°$. The other angle $90° - 45.6° = 44.4°$.