Work by Integration

1. **Finding the work required to stretch a spring**
2. **Finding the work required to wind a wire around a drum**
3. **Finding the work required to pump liquid from a tank**
4. **Finding the work required to move two particles**

Work is defined as the amount of energy required to perform a physical task. When force is constant, work can simply be calculated using the equation

$$W = F \cdot d$$

where W is work, F is a constant force, and d is the distance through which the force acts. The units of work are commonly Newton-meters, Nm; Joules, J; or foot-pound, ft-lb. Frequently, the force is not constant and will change over time. In order to solve for work with a variable force, the following integral equation must be used

$$W = \int_{x=a}^{x=b} f(x)dx$$

where W is work, f(x) is force as a function of distance, and x is distance.

1. Finding the work required to stretch a spring

If an ideal spring is stretched or compressed x units beyond its natural length, then Hooke's Law, $\vec{f}(x) = k\vec{x}$ tells us the force the spring is exerting to resist that action. The proportionality constant k depends on the stiffness of the spring and is determined through empirical testing.

Example 1:

A spring has a natural length of 1 meter. A force of 25 Newtons stretches the spring by ¼ of a meter. Determine how much work is done by stretching the spring.

a) *2 meters beyond its natural length*
b) *From a length of 1.5 meters to 2.5 meters*

We first determine the spring constant, k. Because the force is 25 N when *x* is 0.25 m, we can use Hooke's law to determine *k*.

$$f(x) = kx$$

$$25\ N = k\left(\frac{1}{4}\ m\right)$$

$$\therefore k = 100\ \frac{N}{m} \text{ and } f(x) = 100x$$

Hence, to find the work done by stretching the spring from its rest position to 2 meters beyond that resting position, we do the following:

$$W = \int_a^b f(x)dx = \int_0^2 100x dx = \frac{100}{2}x^2\Big|_0^2 = 50(2)^2 - 50(0)^2 = 200\ Nm \text{ or } 200\ J$$

Similarly, to find the work done by stretching the spring from a length of 1.5 m to 2.5 m:

$$W = \int_a^b f(x)dx = \int_{0.5}^{1.5} 100x dx = \frac{100}{2}x^2\Big|_{0.5}^{1.5} = 50(1.5)^2 - 50(0.5)^2 = 100\ Nm \text{ or } 100\ J$$

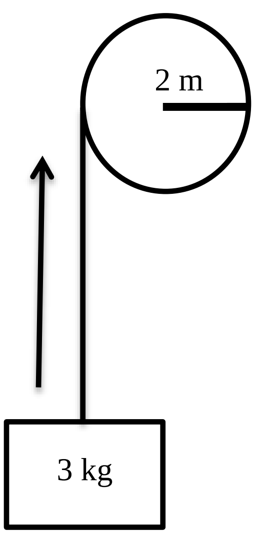
Note: In this type of problem, we need to pay attention to boundaries. If the spring is not stretched, no matter what the length of it is, the lower boundary must be zero. If the spring is stretched to a certain length, then we need to subtract the natural length of it from that value. This then will determine the appropriate boundaries of the integral.

2. Finding the work required to wind a wire around a drum

When winding up a wire, a force has to be applied to pull the mass up against gravity. Therefore, gravitational weight of the mass is equal to the required force to wind the wire. The potential energy gained by the mass is the same as the work done by winding up the wire.

Example 2:

Find the work generated from one revolution of the pictured massless pulley and massless wire system. The mass of the block is 3 kg.



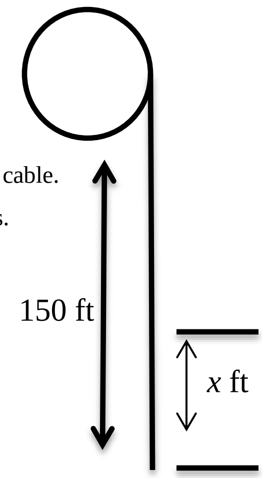
For one rotational revolution of the pulley, the distance the weight is raised is equal to one circumference; therefore, $2\pi r = 2\pi \cdot 2 = 4\pi$. The gravitational force on 3 kg is determined by $mg = (3\ kg) \cdot \left(9.8\ \frac{m}{s^2}\right) = 29.4\ N$. Therefore,

$$W = \int_0^{4\pi} 3 \cdot 9.8\, dx = 29.4 \int_0^{4\pi} dx = 29.4x\Big|_0^{4\pi} = 29.4 \cdot 4\pi - 29.4 \cdot 0 = 369\ J$$

(rounded to 3 significant digits)

Example 3:

We are given a fully extended cable of 150 feet weighing 2.00 lb/ft.
How much work is done after winding 50 feet of cable?



$F = 2\frac{lb}{ft}(150 - x)ft$ [Why (150 – x)? At the beginning there will be 150 feet of hanging cable. As the cable is wound up (by *x* feet) the cable becomes shorter and shorter, weighing less and less. We stop after x = 50 feet are wound, i.e., when there is only 100 feet of cable left hanging.]

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$W = \int_0^{50} 2(150 - x)dx = \int_0^{50}(300 - 2x)dx = [300x - x^2]\,_0^{50} = 300 \cdot 50 - 50^2 =$
$1.25 \text{ x } 10^4 ft \cdot lb$

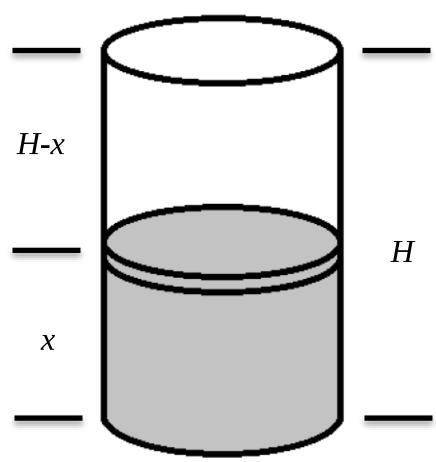
3. Finding work required to pump liquid from a tank

Pumping liquid out of the top of a tank requires work because the liquid is moving against gravity. To calculate this, we imagine the work required to lift small disks of liquid up and out of the tank. So we are lifting a series of masses against gravity and allowing the liquid to spill out once the top is reached. We are asked to calculate the work performed in all of this activity, noting that *g* is the gravitational constant, $32\frac{ft}{s^2}$ or $9.8\frac{m}{s^2}$. Sometimes the density is provided and sometimes the mass and volume are, requiring density to be calculated from this data. What varies in these systems is the distance each disc needs to be lifted, measured by taking the total height, *H*, and subtracting from this the present height of the remaining liquid, *x*, and the volume of each disc.

Example 4:

Find the work done by pumping out water from the top of a cylindrical tank 3.00 *ft in radius and* 10 *ft tall, if the tank is initially full. (The density of water is* $62.4\frac{lb}{ft^3}$*)*

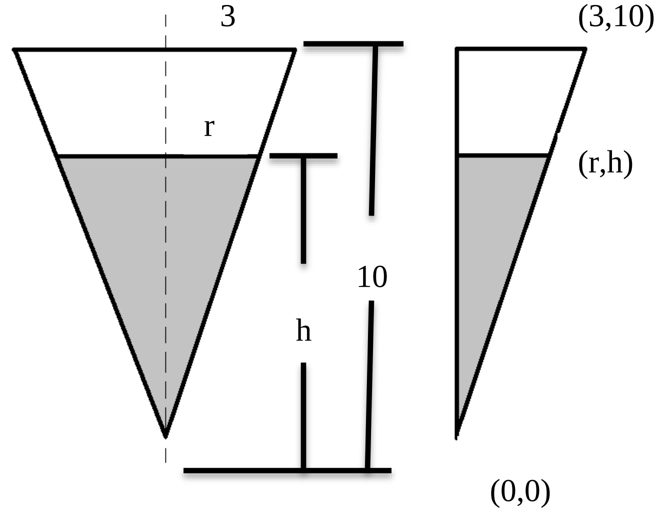
$$F = [Density] \cdot [gravity] \cdot [Area\ of\ Disc] \cdot [Height\ of\ Disc]$$
$$= [\rho] \cdot [g] \cdot [\pi r^2] \cdot [dx]$$



Since *Work* = (*F*·distance) and the distance is *H* – *x*, then

$$W = \int_0^{10} (\rho)(g)(\pi r^2)(H - x)dx$$
$$= \int_0^{10} (62.4)(32)(\pi \cdot 3^2)(10 - x)dx$$
$$= 17971.2\pi \int_0^{10} (10 - x)dx$$
$$= 17971.2\pi \left[10x - \frac{1}{2}x^2\right]_0^{10}$$
$$= 17971.2\pi \left[100 - \frac{1}{2}(10^2)\right] = 2.82 \cdot 10^6 ft \cdot lb$$

Example 5:



Find the work done by pumping out molasses from a conical tank filled to 2 *ft from the top of the tank. The tank has a maximum radius of* 3 *ft and a height of* 10 *ft. Molasses have density* $100\frac{lb}{ft^3}$.

The added difficulty here results from the fact that as the height through which each disc is lifted changes, so does the radius change. We would prefer to integrate with respect to a single variable; therefore, we seek a relationship between the variables h and r.

Using similar triangles:

$$\frac{h}{r} = \frac{10}{3}$$

$$r = \frac{3}{10}h$$

$$F = [Desnity] \cdot [Gravity] \cdot [Area\ of\ Disc] \cdot [Height\ of\ Disc] = [\rho] \cdot [\pi r^2] \cdot [dh]$$

$$W = \int_0^8 (\rho)(g)(\pi r^2)(H-h)dh = \int_0^8 (100)(32)\left(\pi\left(\frac{3}{10}h\right)^2\right)(10-h)dh$$

$$= 288\pi \int_0^8 h^2(10-h)dh = 288\pi \int_0^8 (10h^2 - h^3)dh = 288\pi\left[\frac{10h^3}{3} - \frac{h^4}{4}\right]_0^8$$

$$= 288\pi\left[\frac{10 \cdot 8^3}{3} - \frac{8^4}{4}\right] = 6.176 \text{ x } 10^5 ft \cdot lb$$

You Try It:

Problem 1:

A spring has a natural length of 250 cm. A force of 18 Newtons stretches the spring to a length of 5 meters. Determine how much work is done by stretching the spring

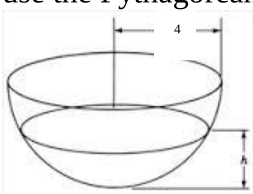
c) *150 cm beyond its natural length*
d) *from a length of 3.5 meters to 5.0 meters*

Problem 2:

Find the work done winding 10 feet of a 25-ft cable that weighs 4.00 lb/ft when there is a 50 lb weight that hangs on the end.

Problem 3:

Find the work done by pumping out molasses from a hemispherical tank with a radius of 4 feet when the initial depth of the molasses is at 2 feet. Molasses has a density of $100\,\frac{lb}{ft^3}$. (Hint: First use the Pythagorean Theorem to show how the relationship between r and h is $r^2 = 8h - h^2$.)



"You Try It" Solutions:

Problem 1: (a) 8.1 Nm; (b) 18.9 Nm

Problem 2: $1300\ ft \cdot lb$

Problem 3: $174{,}253.6725$ *or* $1.74 \times 10^5\ ft \cdot lb$

Sorting out the difference, in the US Customary system of measurement, between pound *mass* and pounds *force* can be challenging. Here is a link to a handout found online from Durham College's Student Academic Learning Services: Pounds Mass vs Pound Force