Preface

An essential role of applied mathematicians is to provide solutions to problems that describe our physical world. In this role, mathematicians encounter a vast array of linear and nonlinear ordinary differential equations (ODEs), for which a range of solution techniques may be applicable. The particular choice of technique for a given problem is often based on the accuracy required for the intended application as well as ease-of-use. Nowadays, as numerical solutions become increasingly embedded within computational software packages, relatively basic training in linear algebra and ODEs prepares one to implement straightforward numerical solutions to a wide array of problems. With today's computational speeds, such solutions are often sufficient. That same numerical solution, however, may become prohibitive when implemented several times as a subroutine within a loop, when used to approximate a solution on an infinite or semi-infinite domain, and/or when further calculus is applied to the output data. Clearly, recognizing that a solution to an ODE may be obtained exactly and analytically alleviates such issues when they naturally arise. Unfortunately, exact analytical solution techniques for nonlinear ODEs are restricted to very specific equation types and are relatively few in number.

It is in the spirit of the above comments that we write this book on power series solutions to nonlinear ODEs. The power series solution technique is known to provide analytical solutions for all classifications of ODEs (including nonlinear) in the interval for which the series converges. That said, there are two main reasons for which chapters on "Power Series Solutions" in an undergraduate differential equations textbook seldom include examples of the method applied to nonlinear ODEs. (1) Convergence is a key limitation of power series solutions. For linear ODEs, the possible locations of convergence-limiting singularities are always known *a priori* as they correspond to singularities in the coefficients and forcing functions, and divergence can be anticipated at the onset. For nonlinear ODEs, convergence-limiting singularities (of the solution) do not typically appear explictly in the ODEs themselves. Thus, one cannot anticipate where the power series solution will fail by simply examining the ODE *as written*. (2) Additionally, nonlinear operations applied to power series lead to (on the surface) messy relationships between unknown power series coefficients that appear difficult to resolve. For example, in R.P. Agnew's book "Differential Equations" [4], an attempt is made to apply the power series method to solve the nonlinear pendulum equation. After several successive applications of the chain rule to determine coefficients of the power series, the number of operations telescope and he writes, "From this point on, complications increase very rapidly and about two more differentiations tempt us to feed the wastebasket and go to work on the next chapter." A chapter is devoted to this exercise in Z. C. Motteler's "Introduction to Ordinary Differential Equations" [76] where it is described as a "gruelling process to calculate further terms". The experiences of Agnew and Motteler are reinforced even by brute force implementation of power series in mathematical software packages. Anyone that has tried to compute coefficients of the power series of a nonlinear ODE using such packages knows that, if one tries to resolve more than a relatively moderate number of terms, the computer fan kicks into high gear while wait times increase astronomically. Although we cannot speak for all software packages, it is quite likely that the implementation of the chain rule employed in Agnew's book is still being used in one form or another and, although not as slow when implemented on a modern computer, is still too slow to render power series solutions as competitors to those obtained using standard numerical methods.

That said, identities for efficiently applying nonlinear operations to power series have long existed in the literature that avoid the above limitations [49]. The authors have utilized these operations—and developed new ones—to solve a wide range of physical problems. In particular, we have successfully obtained analytical solutions of nonlinear differential equations arising in fluid mechanics, astrophysics, general physics, and epidemiology. The results we have obtained are in some cases superior to numerical solutions, as power series coefficients are straightforward to determine recursively, the number of terms needed to represent the solution accurately are small, and the time to obtain an accurate solution is reduced drastically. In short, the solutions are useful and can be implemented even in simple software packages such as spreadsheets.

Our goal, then, in writing this book, is to provide a systematic methodology and guide to the mathematics community for solving nonlinear ODEs via power series. The methodology can also be applied to the solution of certain transcendental equations, as well as linear ODEs having transcendental coefficients that may be expressed as infinite series. To our knowledge, there is no comprehensive book on the subject, likely because of the above-cited limitations. In this book, we provide all the tools to eliminate the need for the laborious calculations that involve manipulation of infinite power series, allowing one to obtain all terms of a power series recursively. We also provide a structured methodology to use series resummations and approximants to overcome convergence barriers that naturally arise. In total, then, we attempt to make the power series solution technique for ODEs both accessible and useful, so that it may augment the applied mathematician's toolbox with available techniques customized for end use.

We have structured the book such that its content may be taught in a single semester course. Necessary course prerequisites are a knowledge of differential equations (analytical and numerical methods), linear algebra, and complex variables. The presentation style throughout the book reflects our teaching philosophy. We believe that mathematics is learned by doing. For that reason, you will find that most of this book is composed of idea-driven examples. In Chapter 1, we provide a list of representative real-world problems that are revisited throughout the book. From there on and until the end of the book, examples build upon each other in levels of increasing complexity. In Chapter 2, we provide the underlying fundamental methodologies needed to implement the power series solution method. The (mostly linear) examples in Chapter 2 are meant to provide the reader with a level of comfort with these methodologies prior to introducing the additional layer of nonlinear series operations in Chapter 3. The examples in Chapter 3 are, in turn, meant to build the reader's skill in using such operations to overcome series divergence in Chapter 4. All theorems and definitions that are needed are contained within, but most proofs are done elsewhere with references provided. We only present such proofs when we anticipate that readers may need to generate their own custom-fit theorem or definition for problems they are likely to encounter in practice. In this case, we show all steps so as to provide guidance for similar derivations. Out of the 150 individual exercise problems in the book (provided at the end of each chapter), nearly half of these either contain a solution to work towards or provide guidance on how to verify that the solution is correct. Since the writing of recursive solutions can differ drastically depending on one's preferred approach and simplification style, we find it best to always check analytical solutions against a converged numerical solution.

In closing, we do not suggest that the power series approach should replace numerical solution methods in all applications where they apply. Rather, we propose that power series solutions should complement numerical approaches, and should be considered as one possible solution technique to meet the end-use envisioned by an applied mathematician. Many thanks for reading this preface, and we hope we have enticed you to read further!

— SJW and NSB