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Cubes of symmetric designs*

Vedran Krčadinac †

Faculty of Science, University of Zagreb, Bijenička cesta 30, HR-10000 Zagreb, Croatia

Mario Osvin Pavčević

Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, HR-10000 Zagreb, Croatia

Kristijan Tabak

Rochester Institute of Technology, Zagreb Campus, D. T. Gavrana 15, HR-10000 Zagreb, Croatia

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Abstract

We study n-dimensional matrices with $\{0,1\}$-entries (n-cubes) such that all their 2-dimensional slices are incidence matrices of symmetric designs. A known construction of these objects obtained from difference sets is generalized so that the resulting n-cubes may have inequivalent slices. For suitable parameters, they can be transformed into n-dimensional Hadamard matrices with this property. In contrast, previously known constructions of n-dimensional designs all give examples with equivalent slices.

Keywords: Symmetric design, difference set, Hadamard matrix.

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1 Introduction

In 1990, Warwick de Launey [9] introduced a framework for higher-dimensional combinatorial designs of various types. Among others, it encompasses symmetric block designs, Hadamard matrices, and generalisations such as orthogonal designs and weighing matrices. While n-dimensional Hadamard matrices have been studied before [3, 4, 15, 16, 31, 32, 33, 35, 36, 37] and after [5, 10, 12, 13, 26] de Launey's paper, and are even featured in books

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†Corresponding author.

E-mail addresses: vedran.krcadinac@math.hr (Vedran Krčadinac), mario.pavcevic@fer.hr (Mario Osvin Pavčević), kristijan.tabak@croatia.rit.edu (Kristijan Tabak)

[1, §6], [2, Chapter 10], [11, Chapter 11], [18, Chapter 5], [38], there seem to be no works dedicated to n-dimensional symmetric designs.

In this paper we study such objects under the name *cubes of symmetric designs*. These are n-dimensional $\{0,1\}$-matrices of order v such that all 2-dimensional slices are incidence matrices of (v,k,λ) designs. Given an ordinary Hadamard matrix of order v, a proper n-dimensional Hadamard matrix of the same order is obtained by the product construction of Yang [35]. Thus, the spectrum of orders v such that proper Hadamard matrices exist is the same for all dimensions $n \geq 2$. According to the famous Hadamard conjecture, it includes all orders divisible by 4. No such construction is known for symmetric designs. There are parameters such that symmetric designs exist, but n-dimensional cubes are not known for $n \geq 3$, for example $(25,9,3)$. However, cubes of arbitrary dimension arise from (v,k,λ) difference sets, in analogy with the group development construction of Hammer and Seberry [15]. We generalise this construction to commence from symmetric designs that have difference sets as blocks, but are not necessarily developments. The ensuing *group cubes* have the interesting property that different slices may be inequivalent (v,k,λ) designs. In contrast, all known constructions of proper n-dimensional Hadamard matrices and other types of n-dimensional designs give examples with equivalent slices.

The outline of our paper is as follows. In Section 2 we recall the relevant definitions and fix notation. Equivalence of cubes and their autotopy and autoparatopy groups are defined. Invariants are introduced to be used later on to distinguish inequivalent cubes. Representations of cubes as orthogonal arrays and transversal designs are explained, enabling the application of computational tools for incidence structures to cubes of symmetric designs.

Section 3 is devoted to the construction of n-cubes from difference sets. Properties of such *difference cubes* and their autotopy groups are proved. Equivalence of difference sets and the corresponding difference cubes is compared. Distinctions are found between dimension $n = 2$ and dimensions $n \geq 3$.

The group cube construction is introduced in Section 4. Examples not equivalent to difference cubes are given for parameters $(21,5,1)$, $(16,6,2)$, and extended to $(4^m, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$ in Theorem 4.3. These are parameters of Menon designs; thus, these cubes can be transformed into proper n-dimensional Hadamard matrices with inequivalent slices. Equivalence of n-dimensional Hadamard matrices was studied in [26] and [13]. Interestingly, it turns out that the product construction of [35] gives many inequivalent n-dimensional Hadamard matrices from a single 2-dimensional matrix H. However, all slices of the n-dimensional product matrix are equivalent with H, and hence also mutually equivalent. Our n-dimensional Hadamard matrices with inequivalent slices can therefore not be obtained by the product construction, nor by other previously known methods.

Finally, Section 5 is devoted to computational results about cubes with small parameters. A complete classification of group cubes is performed for $(16,6,2)$ and $(21,5,1)$. Examples of non-difference group cubes are found for $(27,13,6)$, $(36,15,6)$, $(45,12,3)$, $(63,31,15)$, $(64,28,12)$, and $(96,20,4)$. More examples of $(16,6,2)$ cubes are constructed by prescribing autotopy groups. Some of them have inequivalent parallel slices in all directions, and are therefore not equivalent with any group cube.

Applications of n-dimensional Hadamard matrices include spectroscopy, error-correcting codes, cryptography, and signal processing; see [18, Section 5.3] and [38]. Cubes of symmetric designs could have similar applications, but this will be left for future works.

2 Definitions

Let V be a set of v points. A (v,k,λ) *design* over V is a collection $\mathcal{D}$ of k-subsets of V called *blocks*, such that every pair of points is contained in exactly λ blocks. The design is *symmetric* if the number of blocks is also v. All designs in this paper are symmetric. Given an ordering of the points $p_1,\ldots,p_v$ and blocks $B_1,\ldots,B_v$, the *incidence matrix* $A=(a_{ij})$ of the design is defined by $a_{ij}=[p_i\in B_j]$. Here and in the sequel, we use the Iverson bracket $[P]$ which takes the value 1 if P is true and 0 otherwise; see [21]. It is known that a $v\times v$ matrix A with $\{0,1\}$-entries is the incidence matrix of a (v,k,λ) design if and only if $A\cdot A^t=(k-\lambda)I+\lambda J$ holds, where I is the identity matrix, and J is the all-ones matrix. Furthermore, the transposed matrix A^t satisfies the same equation and is an incidence matrix of the dual design $\mathcal{D}^t$. We refer to [25] for these and other results about symmetric designs.

Let $\{1,\ldots,v\}^n$ be the Cartesian product of n copies of $\{1,\ldots,v\}$. An n-dimensional *incidence cube* of order v is a function $C:\{1,\ldots,v\}^n\to\{0,1\}$. We will talk about n-cubes of order v for short. Thus, a 2-cube is a $v\times v$ matrix with $\{0,1\}$-entries, a 3-cube is a 3-dimensional array of zeros and ones, etc. Given a pair of distinct integers $(x,y)\in\{1,\ldots,n\}^2$, a *slice* of the n-cube C is the matrix obtained by varying the coordinates in positions x and y, and taking some fixed values $i_1,\ldots,i_{x-1},i_{x+1},\ldots,i_{y-1},i_{y+1},\ldots,i_n\in\{1,\ldots,v\}$ for the remaining coordinates. In other words, it is the restriction of C to the set

$$\{i_1\}\times\cdots\times\{i_{x-1}\}\times V\times\{i_{x+1}\}\times\cdots\times\{i_{y-1}\}\times V\times\{i_{y+1}\}\times\cdots\times\{i_n\}.$$

We allow $x>y$, in which case the order of the factors indexed by x and y should be reversed. The slices corresponding to (x,y) and (y,x) are transposed matrices.

Definition 2.1. An *n-dimensional cube of symmetric* (v,k,λ) *designs* is an n-cube of order v such that all of its slices are incidence matrices of (v,k,λ) designs. The set of all such n-cubes will be denoted by $\mathcal{C}^n(v,k,\lambda)$.

This is a special case of de Launey's proper n-dimensional transposable designs $(v,\Pi_R,\Pi_C,\beta,S)^n$, see [9, Definitions 2.1, 2.6, 2.7, and Example 2.2]. Here is a concrete example of such an object.

Example 2.2. Consider the following incidence matrix of the Fano plane, i.e. the $(7,3,1)$ design:

$$A_1=\begin{pmatrix}1&1&0&1&0&0&0\\1&0&1&0&0&0&1\\0&1&0&0&0&1&1\\1&0&0&0&1&1&0\\0&0&0&1&1&0&1\\0&0&1&1&0&1&0\\0&1&1&0&1&0&0\end{pmatrix}.$$

Let A_2 be the matrix obtained by a cyclic shift upwards of rows of A_1. This is another incidence matrix of the Fano plane. Continue in the same way to get incidence matrices A_3 to A_7:

$$A_2 = \begin{pmatrix} 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0 \end{pmatrix},$$

$$A_6 = \begin{pmatrix} 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1 \end{pmatrix}, \quad A_7 = \begin{pmatrix} 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0 \end{pmatrix}.$$

Now put all of these matrices on top of each other: first A_1, then A_2 and end up with A_7 as the top layer. We get a 3-cube of order 7 such that the horizontal slices are incidence matrices of the $(7,3,1)$ design. Because of the cyclic nature of the construction, this also holds for slices in the other two directions. If we look at a particular row of $A_1,\dots,A_7$, say row j, we get the matrix A_j. The matrices are symmetric and the conclusion also holds if we look at a particular column.

We now define equivalence of cubes of symmetric designs; cf. [13, 26] for the corresponding concepts for n-dimensional Hadamard matrices. The direct product of symmetric groups $(S_v)^n = S_v \times \dots \times S_v$ acts on the set $\mathcal{C}^n(v,k,\lambda)$ by permuting indices: for $\alpha = (\alpha_1,\dots,\alpha_n) \in (S_v)^n$,

$$C^\alpha(i_1,\dots,i_n) = C(\alpha_1^{-1}(i_1),\dots,\alpha_n^{-1}(i_n)).$$

The orbits of this action are the *isotopy classes* of cubes. The stabiliser of a cube $C \in \mathcal{C}^n(v,k,\lambda)$ is its *autotopy group* $\mathrm{Atop}(C)$. Furthermore, permutations $\gamma \in S_n$ act by *conjugation*, i.e. changing the order of the indices:

$$C^\gamma(i_1,\dots,i_n) = C(i_{\gamma^{-1}(1)},\dots,i_{\gamma^{-1}(n)}).$$

A cube such that $C^\gamma = C$ for all $\gamma \in S_n$ is *totally symmetric*. The cube from Example 2.2 has this property. The combination of isotopy and conjugation is called *paratopy* and is the natural action of the wreath product $S_v \wr S_n$ on $\mathcal{C}^n(v,k,\lambda)$. Two cubes are considered *equivalent* if they can be mapped onto each other by paratopy. The corresponding classes are the *main classes* and the stabiliser of a cube C in $S_v \wr S_n$ is its *autoparatopy group* $\mathrm{Apar}(C)$.

The terminology is borrowed from Latin squares [20]. Indeed, a Latin square $L = (\ell_{i_1 i_2})$ of order v is equivalent to a 3-cube $C \in \mathcal{C}^3(v,1,0)$ by $C(i_1,i_2,i_3) = [\ell_{i_1 i_2} = i_3]$, since incidence matrices of symmetric designs with $k=1$ are just permutation matrices. The equivalence is easily generalised to higher dimensions, but notice that different authors have considered different concepts under the name "Latin hypercubes". The appropriate definition for our purposes is the one used in [29]; see the discussion and references therein for other related concepts. In the case $k=2$, symmetric designs are possible only for parameters $(3,2,1)$. There is a unique $(3,2,1)$ design up to isomorphism and it is cyclic. By Theorem 3.1, cubes of $(3,2,1)$ designs exist for all dimensions $n \geq 2$. In the sequel we consider only designs with $k \geq 3$.

To distinguish inequivalent cubes, we use invariants based on parallel slices. For dimension $n=2$, cubes are simply incidence matrices of symmetric designs and equivalence means that the designs are isomorphic or dually isomorphic. For $n>2$, we can choose the varying coordinates in $\binom{n}{2}$ ways and substitute the fixed coordinates in v^{n-2} ways. Slices are *parallel* if all but one of the fixed coordinates agree. We consider v mutually parallel slices as a multiset of (dual) isomorphism types of (v,k,λ) designs. The multiset of $\binom{n}{2}(n-2)v^{n-3}$ such multisets must agree for equivalent cubes, i.e. this is a paratopy invariant. We call it the *slice invariant* of the cube. We may also look at weaker invariants, for example orders of the full automorphism groups of parallel slices. In either case, the multiset of such multisets agreeing is only a necessary, but not sufficient condition for equivalence of cubes; see the comment after Example 4.2.

An incidence cube $C\colon \{1,\ldots,v\}^n \to \{0,1\}$ is the characteristic function of a set of n-tuples

$$\overline{C} = \{(i_1,\ldots,i_n) \in \{1,\ldots,v\}^n \mid C(i_1,\ldots,i_n) = 1\}.$$

This is a one-to-one correspondence. If $C \in \mathcal{C}^n(v,k,\lambda)$, then $\overline{C}$ is an orthogonal array with parameters $OA(kv^{n-1},n,v,n-1)$ of index k; see [17] or [8, Section III.6] for the definition. It has the additional property that for any choice of coordinate positions $x,y \in \{1,\ldots,n\}$, $x \neq y$, and values $i_1,\ldots,i_{x-1},i_{x+1},\ldots,i_{y-1},i'_y,i''_y,i_{y+1},\ldots,i_n \in \{1,\ldots,v\}$, $i'_y \neq i''_y$, there are exactly λ values $i \in \{1,\ldots,v\}$ such that $(i_1,\ldots,i_{x-1}, i,i_{x+1},\ldots,i'_y,\ldots,i_n) \in \overline{C}$ and $(i_1,\ldots,i_{x-1},i,i_{x+1},\ldots,i''_y,\ldots,i_n) \in \overline{C}$. Every orthogonal array with this property and the parameters above corresponds to an n-dimensional cube of (v,k,λ) designs. Thus, we can represent cubes of symmetric designs by orthogonal arrays. In practice we use an equivalent representation by transversal designs: $\overline{C}$ is represented as a collection of n-subsets $\{i_1, v+i_2, 2v+i_3,\ldots,(n-1)v+i_n\}$ instead of n-tuples $(i_1,\ldots,i_n)$. By *transversal design*, here we mean a set of nv points partitioned into n "groups" of size v, together with a collection of blocks, each containing exactly one point from every group (i.e. blocks are transversals of the groups). Transversal designs are normally used to represent sets of mutually orthogonal Latin squares (see [8, Section III.3]).

Now we have a representation of cubes by incidence structures of nv points and kv^{n-1} blocks such that paratopy of cubes agrees with the usual notion of isomorphism of incidence structures. This has the advantage that we can check equivalence of cubes and compute autoparatopy groups by tools available in the computer algebra system GAP [14], notably the graph isomorphism programs *nauty* and *Traces* [28]. By coloring the n groups of points $\{1, \ldots, v\}, \ldots, \{(n-1)v+1, \ldots, nv\}$ we can also check isotopy of cubes, compute autotopy groups, and use the Kramer-Mesner approach [22] to construct cubes with prescribed autotopy groups. These and other tools for handling cubes of symmetric designs are available in our GAP package *Prescribed Automorphism Groups* [24].

3 Difference cubes

Let G be a multiplicatively written group of order v. A (v, k, λ) *difference set* in G is a k-subset $D \subseteq G$ such that every $g \in G$, $g \neq 1$ can be written as a left difference $g = d_1^{-1} d_2$ for exactly λ pairs $(d_1, d_2) \in D \times D$. This is equivalent with the corresponding property for right differences $g = d_1 d_2^{-1}$ [7]. The *development* of D, i.e. the set of left translates $\operatorname{dev} D = \{gD \mid g \in G\}$ is a (v, k, λ) design over G. The right translates $\{Dg \mid g \in G\}$ also form a design isomorphic to the dual $(\operatorname{dev} D)^t$. Difference sets give rise to cubes of symmetric designs of arbitrary dimension. The following theorem is a special case of [9, Theorem 2.9]. An analogous theorem was originally proved for group developed n-dimensional proper Hadamard matrices [15, Theorem 4].

Theorem 3.1. *Let D be a (v, k, λ) difference set in the group G. Order the group elements as $g_1, \ldots, g_v$. Then the function*

$$C(i_1, \ldots, i_n) = [g_{i_1} \cdots g_{i_n} \in D] \tag{3.1}$$

is an n-dimensional cube of (v, k, λ) designs.

Proof. First consider the case $n = 2$. Notice that the design $\operatorname{dev} D$ with points ordered as $g_1, \ldots, g_v$ and blocks ordered as $g_1 D, \ldots, g_v D$ has incidence matrix $A = (a_{ij})$ given by $a_{ij} = [g_i \in g_j D]$. The matrix $C(i, j) = [g_i g_j \in D]$ is an incidence matrix of the dual design $(\operatorname{dev} D)^t$ with a different ordering of columns, since $g_i g_j \in D \iff g_j \in g_i^{-1} D$ holds. In the general case, a slice of the n-cube (3.1) is a matrix of the form $a_{ij} = [h_1 g_i h_2 g_j h_3 \in D]$ or $a_{ij} = [h_1 g_j h_2 g_i h_3 \in D]$ for some fixed $h_1, h_2, h_3 \in G$. This is just a permutation of rows and columns of the incidence matrix of $(\operatorname{dev} D)^t$ or $\operatorname{dev} D$, and therefore (3.1) is a (v, k, λ) cube of dimension n. □

The general construction of Theorem 2.9 in [9] relies on *collapsable functions*, which were later specialised to *abelian extension functions* and 2-*cocycles* [12]. Cocyclic Hadamard matrices are an important and widely studied class [18], more general than group developed Hadamard matrices. The latter are necessarily regular, and therefore of order $v = 4u^2$ for some $u \in \mathbb{N}$. There is no such restriction for cocyclic Hadamard matrices, which are conjectured to exist for all orders divisible by 4 [12, Conjecture 3.6]. Regarding symmetric designs, the symbols 0 and 1 in their incidence matrices cannot be exchanged and there are no nontrivial cocycles. In this case Theorem 2.9 of [9] reduces to the difference set construction of Theorem 3.1.

We shall call cubes arising from Theorem 3.1 *difference cubes*. A *non-difference cube* is a cube of symmetric designs not equivalent to any difference cube. The following properties of difference cubes are easy consequences of the definition.

Proposition 3.2. *All slices of an n-dimensional difference cube are incidence matrices of the same (v, k, λ) design up to isomorphism and duality.*

Proposition 3.3. *A difference cube constructed from an abelian group is totally symmetric, i.e. invariant under any conjugation.*

A design $\operatorname{dev} D$ constructed from a difference set $D \subseteq G$ has G as an automorphism group acting regularly on the points and blocks. An n-dimensional difference cube has the direct product $G^{n-1} = G \times \ldots \times G$ as autotopy group acting as follows. For $a, b \in G$, let ${}_a\alpha_b \in S_v$ be the permutation defined by ${}_a\alpha_b(i) = j \iff a g_i b^{-1} = g_j$. Given $(a_1, \ldots, a_{n-1}) \in G^{n-1}$, let $a_0 = a_n = 1$ and $\alpha_i = {}_{a_{i-1}}\alpha_{a_i}$, for $i = 1, \ldots, n$. Then $(a_1, \ldots, a_{n-1}) \mapsto (\alpha_1, \ldots, \alpha_n) =: \alpha$ is a group embedding $G^{n-1} \hookrightarrow (S_v)^n$ and its image is an autotopy group of the difference cube (3.1):

$$
\begin{aligned}
C(i_1, \ldots, i_n) &= [g_{i_1} \cdots g_{i_n} \in D] \\
&= [g_{i_1} a_1 \cdot a_1^{-1} g_{i_2} a_2 \cdot \ldots \cdot a_{n-2}^{-1} g_{i_{n-1}} a_{n-1} \cdot a_{n-1}^{-1} g_{i_n} \in D] \\
&= [g_{\alpha_1^{-1}(i_1)} \cdots g_{\alpha_n^{-1}(i_n)} \in D] = C^{\alpha}(i_1, \ldots, i_n).
\end{aligned}
$$

A *multiplier* of the difference set $D \subseteq G$ is a group automorphism $\varphi \colon G \to G$ such that $\varphi(D) = aD$ for some $a \in G$. The set of all multipliers of D is a subgroup $\operatorname{Mult}(D) \leq \operatorname{Aut}(G)$ acting on the design $\operatorname{dev} D$ as automorphism group. This group also acts on the difference cube (3.1) as autotopy group:

$$
\begin{aligned}
g_{i_1} \cdots g_{i_n} \in D \iff \varphi(g_{i_1} \cdots g_{i_n}) \in \varphi(D) &\iff \varphi(g_{i_1}) \cdots \varphi(g_{i_n}) \in aD \\
&\iff a^{-1} \varphi(g_{i_1}) \cdots \varphi(g_{i_n}) \in D.
\end{aligned}
$$

In the last line we see how to permute the indices to leave the difference cube (3.1) invariant. The combined actions of G^{n-1} and $\operatorname{Mult}(D)$ give an autotopy group of (3.1) isomorphic to their semidirect product. This proves the following theorem.

Theorem 3.4. *The difference cube* (3.1) *has an autotopy group isomorphic to a semidirect product* $(G^{n-1}) \rtimes \operatorname{Mult}(D)$.

Two difference sets $D_1 \subseteq G_1$ and $D_2 \subseteq G_2$ are *equivalent* if there is a group isomorphism $\varphi \colon G_1 \to G_2$ such that $\varphi(D_1) = aD_2$ for some $a \in G_2$. Equivalent difference sets clearly give rise to isomorphic designs and isotopic difference cubes. For designs, the converse does not hold: $\operatorname{dev} D_1$ and $\operatorname{dev} D_2$ can be isomorphic even if D_1 and D_2 are not equivalent. For example, according to the GAP package *DifSets* [30] there are 27 inequivalent $(16, 6, 2)$ difference sets in 12 of the 14 groups of order 16 (see Table 1). On the other hand, there are only three $(16, 6, 2)$ designs up to isomorphism and duality [8]. By a computation in PAG [24], the 27 difference 3-cubes $(16, 6, 2)$ are not equivalent (paratopic). The conclusion holds for all dimensions $n \geq 3$ because any sub-3-cube of a difference n-cube is equivalent with the corresponding difference 3-cube.

The smallest examples of equivalent difference n-cubes ($n \geq 3$) obtained from inequivalent difference sets have parameters $(27, 13, 6)$. Difference sets exist in two of the five groups of order 27. There is a unique $(27, 13, 6)$ difference set in $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and two inequivalent difference sets in $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ [30]. Another computation in PAG [24] shows that the difference 3- and 4-cubes constructed from the two difference sets in $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ are equivalent, but not isotopic. See Table 3 for other examples of equivalent difference cubes coming from inequivalent difference sets. For dimensions $n \geq 3$, we did not find a single example of isotopic difference cubes coming from inequivalent difference sets, nor of equivalent difference cubes coming from difference sets in nonisomorphic groups. Both situations are possible for $n = 2$.

4 Group cubes

In this section we generalise the construction of difference cubes of Theorem 3.1. Let $G = \{g_1, \ldots, g_v\}$ be a group of order v and $\mathcal{D} = \{B_1, \ldots, B_v\}$ a (v, k, λ) design over G such that all blocks B_i are (v, k, λ) difference sets. The design $\mathcal{D}$ could be the development of a difference set, say $B_i = g_i^{-1} B_1$, $i = 1, \ldots, v$. In this case the corresponding difference cube can be written as $C(i_1, \ldots, i_n) = [g_{i_1} \cdots g_{i_n} \in B_1] = [g_{i_2} \cdots g_{i_n} \in g_{i_1}^{-1} B_1] = [g_{i_2} \cdots g_{i_n} \in B_{i_1}]$. The last formula gives a (v, k, λ) cube even if $\mathcal{D}$ is not a development.

Theorem 4.1. *Let $G = \{g_1, \ldots, g_v\}$ be a group and $\mathcal{D} = \{B_1, \ldots, B_v\}$ a (v, k, λ) design with all of its blocks being (v, k, λ) difference sets in G. Then*

$$C(i_1, \ldots, i_n) = [g_{i_2} \cdots g_{i_n} \in B_{i_1}] \tag{4.1}$$

is an n-dimensional cube of (v, k, λ) designs.

Proof. If the index i_1 is fixed, then (4.1) is just the $(n-1)$-dimensional difference cube constructed from the difference set B_{i_1}, so any such slice is an incidence matrix of its development. On the other hand, if i_1 and one of the remaining indices are varied, then the corresponding slice is an incidence matrix of the design $\mathcal{D}^t$ with reordered columns. □

We shall call cubes constructed from Theorem 4.1 *group cubes*. The main question is whether there exist designs that are not developments, but all of their blocks are difference sets. In this case the construction may give non-difference cubes. Our first example is for parameters $(21, 5, 1)$ of the projective plane of order 4.

Example 4.2. There are two groups of order 21: the Frobenius group $F_{21} = \langle a, b \mid a^3 = b^7 = 1,\ ba = ab^2 \rangle$ and the cyclic group $\mathbb{Z}_{21}$. Both groups allow one $(21, 5, 1)$ difference set up to equivalence [30]. We denote the corresponding 3-cubes C_1 and C_2, respectively. Using PAG [24], we computed $|\operatorname{Atop}(C_1)| = 1323$ and $|\operatorname{Atop}(C_2)| = 2646$. Thus, C_1 and C_2 are not equivalent, and neither are the corresponding n-cubes for $n > 3$ because all of their sub-3-cubes are equivalent either with C_1 or with C_2. This shows that there are exactly two inequivalent difference cubes in $\mathcal{C}^n(21, 5, 1)$ for every $n \geq 3$.

Here is a $(21, 5, 1)$ design over F_{21} with all blocks being difference sets, which is not the development of any of its blocks:

$$\begin{aligned}
\{\ &\{1, a, b, b^3, a^2b^2\},\ \{a^2b^6, b^6, a^2b^3, a^2b^4, a\},\ \{1, a^2, ab, b^2, b^6\},\\
&\{a^2b, ab, b^5, a^2b^2, a^2b^4\},\ \{1, a^2b, a^2b^5, ab^6, a^2b^6\},\ \{ab^6, b, b^2, a^2b^4, b^4\},\\
&\{1, ab^3, b^4, a^2b^3, b^5\},\ \{a^2b^5, b^3, a^2, ab^3, a^2b^4\},\ \{b, a^2, a^2b, a^2b^3, ab^5\},\\
&\{ab^5, b^3, b^5, b^2, a^2b^6\},\ \{b, ab^2, b^5, b^6, a^2b^5\},\ \{a^2, b^4, a^2b^2, a^2b^6, ab^2\},\\
&\{b^2, a^2b^2, a^2b^3, ab^4, a^2b^5\},\ \{ab^4, b^4, b^6, a^2b, b^3\},\ \{1, ab^2, ab^4, a^2b^4, ab^5\},\\
&\{a, a^2, ab^4, b^5, ab^6\},\ \{a, ab, b^4, ab^5, a^2b^5\},\ \{a, b^2, a^2b, ab^2, ab^3\},\\
&\{b, ab, ab^3, ab^4, a^2b^6\},\ \{ab, ab^2, b^3, a^2b^3, ab^6\},\ \{a^2b^2, ab^3, ab^5, b^6, ab^6\}\ \}.
\end{aligned}$$

By Theorem 4.1, the design yields a 3-cube C_3. We computed $|\operatorname{Atop}(C_3)| = 441$, and therefore C_3 is not equivalent neither with C_1 nor with C_2. As before, by considering sub-3-cubes, we see that the conclusion holds for all dimensions: $\mathcal{C}^n(21, 5, 1)$ contains a non-difference group cube for every $n \geq 3$.

The inequivalence of the cubes C_1, C_2, and C_3 was shown by calculating their full autotopy groups. There is only one $(21,5,1)$ design $\mathcal{D}_0$ up to isomorphism and duality, so these three cubes have the same slice invariant $\{\{\mathcal{D}_0^{21}\}^3\}$ defined on page 5. The notation means that in any of the 3 directions, the 21 parallel slices are incidence matrices of the same design $\mathcal{D}_0$. For parameters $(16,6,2)$, there are three inequivalent designs $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ [8] with full automorphism groups of orders $|\operatorname{Aut}(\mathcal{D}_1)| = 11520$, $|\operatorname{Aut}(\mathcal{D}_2)| = 768$, and $|\operatorname{Aut}(\mathcal{D}_3)| = 384$. By Proposition 3.2, the 27 inequivalent difference n-cubes $(16,6,2)$ have slice invariants $\{\{\mathcal{D}_i^{16}\}^e\}$ for $e = \binom{n}{2}(n-2)16^{n-3}$ and $i \in \{1,2,3\}$. Our next goal is to construct group cubes with different slice invariants, which will then clearly be non-difference cubes.

The design $\mathcal{D}_1$ belongs to a family of symmetric designs with the symmetric difference property (*SDP designs*) studied by Kantor [19]. They can be constructed from difference sets in elementary abelian groups $G_m = \mathbb{Z}_2^{2m}$ of orders 4^m as follows. Any singleton, e.g. $D_1 = \{00\}$, is a $(4,1,0)$ difference set in the Klein four-group $G_1 = \{00, 01, 10, 11\}$. Here we use additive notation and 00 is the neutral element. By the product construction of Mann [27, Lemma 6.3.1],

$$D_m = (D_{m-1}^c \times D_1) \cup (D_{m-1} \times D_1^c) \tag{4.2}$$

is a $(4^m, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$ difference set in the group $G_m = G_{m-1} \times G_1$. The development of D_m is an SDP design denoted $\mathscr{S}^{-1}(2m)$ in [19], and the design $\mathcal{D}_1$ coincides with $\mathscr{S}^{-1}(4)$.

The other two $(16,6,2)$ designs $\mathcal{D}_2$ and $\mathcal{D}_3$ can be obtained by switching submatrices in the incidence matrix of $\mathcal{D}_1$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Clearly any such switch leaves the row and column sums invariant. It can be checked that dot products of rows and columns are also not changed. To be specific, number the group elements of $G_2 = \mathbb{Z}_2^4$ lexicographically: $g_0 = 0000$, $g_1 = 0001$, $g_2 = 0010$, $g_3 = 0011, \ldots, g_{15} = 1111$. Let $D_2 = \{g_1, g_2, g_3, g_4, g_8, g_{12}\}$ be the difference set for $\mathcal{D}_1$. Denote the blocks $B_i = g_i + D_2$, so that $\mathcal{D}_1 = \{B_0, \ldots, B_{15}\}$. Then the blocks $B_0 = D_2$ and $B_1 = \{g_0, g_2, g_3, g_5, g_9, g_{13}\}$ contain the elements g_2, g_3 and do not contain the elements g_{14}, g_{15}. On the other hand, the blocks $B_{12} = \{g_0, g_4, g_8, g_{13}, g_{14}, g_{15}\}$ and $B_{13} = \{g_1, g_5, g_9, g_{12}, g_{14}, g_{15}\}$ do not contain g_2, g_3 and do contain g_{14}, g_{15}. By replacing these four blocks by their symmetric difference with $S_1 = \{g_2, g_3, g_{14}, g_{15}\}$ and leaving the other blocks unchanged, the design $\mathcal{D}_1$ is transformed into $\mathcal{D}_2$. Denote the new blocks $B_i' = B_i \Delta S_1$ for $i = 0, 1, 12, 13$ and $B_i' = B_i$ for $i = 2, \ldots, 11, 14, 15$. The design $\mathcal{D}_3$ is obtained by applying another switch to $\mathcal{D}_2$. Take $S_2 = \{g_6, g_7, g_{14}, g_{15}\}$ and replace the blocks $B_i'' = B_i' \Delta S_2$, $i = 0, 1, 4, 5$ leaving the other blocks unchanged. The designs $\mathcal{D}_2 = \{B_0', \ldots, B_{15}'\}$ and $\mathcal{D}_3 = \{B_0'', \ldots, B_{15}''\}$ are not SDP designs.

Now notice that the changed blocks B_i' and B_i'' are also $(16,6,2)$ difference sets in the group G_2, equivalent with the original difference set D_2. Therefore, $\mathcal{D}_2$ and $\mathcal{D}_3$ are $(16,6,2)$ designs with all blocks being difference sets in G_2. The development of any one of these blocks is isomorphic to $\mathcal{D}_1$, thus $\mathcal{D}_2$ and $\mathcal{D}_3$ are not developments of their blocks. By applying Theorem 4.1 to these designs, we get group cubes of dimension $n \geq 3$

with slice invariants $\{\{\mathcal{D}_1^{16}\}^{e_1}, \{\mathcal{D}_2^{16}\}^{e_2}\}$ and $\{\{\mathcal{D}_1^{16}\}^{e_1}, \{\mathcal{D}_3^{16}\}^{e_2}\}$, respectively, for $e_1 = \binom{n-1}{2}(n-2)16^{n-3}$ and $e_2 = (n-1)(n-2)16^{n-3}$.

We now generalise this construction to obtain non-difference cubes in $\mathcal{C}^n(4^m, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$.

Theorem 4.3. *For every $m \geq 2$ and $n \geq 3$, the set $\mathcal{C}^n(4^m, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$ contains at least two inequivalent group cubes that are not difference cubes.*

Proof. Let A be an incidence matrix of a symmetric design with parameters $(4^m, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$ and J the all-ones matrix of order 4^m. Then $J-A$ is the incidence matrix of the complementary design with parameters $(4^m, 2^{m-1}(2^m+1), 2^{m-1}(2^{m-1}+1))$. The block matrix

$$\begin{pmatrix} J-A & A & A & A \\ A & J-A & A & A \\ A & A & J-A & A \\ A & A & A & J-A \end{pmatrix} \tag{4.3}$$

is an incidence matrix of a $(4^{m+1}, 2^m(2^{m+1}-1), 2^m(2^n-1))$ design. This recursive construction was described in [6] and [19] by using $\{-1,1\}$-matrices and the Kronecker product.

When applied to the incidence matrix of $\mathcal{D}_1$, the construction gives the series of SDP designs $\mathscr{S}^{-1}(2m)$. Applied to $\mathcal{D}_2$ and $\mathcal{D}_3$, the construction gives two other series of nonisomorphic designs $\mathscr{S}_2^{-1}(2m)$ and $\mathscr{S}_3^{-1}(2m)$ with the same parameters. The blocks of these designs are $(4^m, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$ difference sets in G_m, because the first block row of (4.3) is obtained by the product construction (4.2) of rows of A with the singleton $\{00\}$, the second block row with the singleton $\{01\}$, etc. The developments of these difference sets are all isomorphic to $\mathscr{S}^{-1}(2m)$, so $\mathscr{S}_2^{-1}(2m)$ and $\mathscr{S}_3^{-1}(2m)$ are not developments of their blocks. Theorem 4.1 gives rise to two group n-cubes in G_m. Slices with i_1 fixed are incidence matrices of $\mathscr{S}^{-1}(2m)$, and slices with i_1 varying are incidence matrices of $\mathscr{S}_2^{-1}(2m)$ or $\mathscr{S}_3^{-1}(2m)$. Therefore, these two n-cubes are not difference cubes and the theorem is proved. □

The exact number of inequivalent group cubes in $\mathcal{C}^n(4^m, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$ is much larger; see Proposition 5.1 for $n=3$ and $m=2$. Notice that the parameters are of Menon type and therefore by exchanging $0 \to -1$ the cubes are transformed into n-dimensional proper Hadamard matrices. These matrices are *totally regular*, meaning that any slice is a regular Hadamard matrix. The designs $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ are transformed into inequivalent Hadamard matrices H_1, H_2, H_3. Up to equivalence, there are only two other Hadamard matrices H_4 and H_5 of order 16 [34]. The series of designs $\mathscr{S}^{-1}(2m)$, $\mathscr{S}_2^{-1}(2m)$, $\mathscr{S}_3^{-1}(2m)$ are transformed into Hadamard matrices which are Kronecker products of H_1, H_2, H_3 with the cyclic Hadamard matrix of order 4, and are also not equivalent. In conclusion, the two non-difference cubes of Theorem 4.3 can be transformed into proper n-dimensional Hadamard matrices with inequivalent slices, which cannot be obtained by [9, Theorem 2.9] and other known constructions such as the product construction of Yang [35].

5 Small examples

The smallest parameters of symmetric designs are $(7,3,1)$, $(11,5,2)$, and $(13,4,1)$. There is a single design up to isomorphism and duality for each of these parameters, coming from a difference set in the cyclic group $\mathbb{Z}_v$. The next parameters are $(15,7,3)$ with four designs up to isomorphism and duality, but only one coming from a difference set, namely $PG_2(3,2)$. In all four cases the only designs with difference sets as blocks are developments. Hence, there is a unique group cube in $\mathcal{C}^n(7,3,1)$, $\mathcal{C}^n(11,5,2)$, $\mathcal{C}^n(13,4,1)$, and $\mathcal{C}^n(15,7,3)$, equivalent with the difference cube.

The fifth smallest parameters are $(16,6,2)$, and in this case there are many non-difference group cubes. The results of an exhaustive computer search for designs with $(16,6,2)$ difference sets as blocks are given in Table 1. The first two columns contain IDs of the groups of order 16 in the GAP library of small groups [14] and a description of their structure. The third and fourth column (Nds and Ndc) contain numbers of inequivalent difference sets according to [30] and numbers of inequivalent difference cubes, respectively. These numbers coincide by the computation from Section 3. The fifth column contains designs arising as developments of difference sets in each of the groups. The total number of difference sets, including equivalent ones, is given in the sixth column (Tds). The last column (Ngc) contains numbers of inequivalent non-difference 3-cubes constructed by Theorem 4.1. For example, there are 58 inequivalent group 3-cubes in $\mathbb{Z}_4^2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, three of which are difference cubes, and 55 of which are non-difference cubes. Here is a summary of the classification.

ID	Structure	Nds	Ndc	dev D	Tds	Ngc
1	$\mathbb{Z}_{16}$	0	0	–	0	0
2	$\mathbb{Z}_4^2$	3	3	$\mathcal{D}_1$	192	55
3	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	4	4	$\mathcal{D}_1$	192	83
4	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	3	3	$\mathcal{D}_1$	192	81
5	$\mathbb{Z}_8 \times \mathbb{Z}_2$	2	2	$\mathcal{D}_1, \mathcal{D}_2$	192	106
6	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	2	2	$\mathcal{D}_1$	64	34
7	D_{16}	0	0	–	0	0
8	QD_{16}	2	2	$\mathcal{D}_1$	128	50
9	Q_{16}	2	2	$\mathcal{D}_1$	256	71
10	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	2	2	$\mathcal{D}_1$	448	131
11	$\mathbb{Z}_2 \times D_8$	2	2	$\mathcal{D}_1$	192	52
12	$\mathbb{Z}_2 \times Q_8$	2	2	$\mathcal{D}_1, \mathcal{D}_3$	704	197
13	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	2	2	$\mathcal{D}_1, \mathcal{D}_3$	320	77
14	$\mathbb{Z}_2^4$	1	1	$\mathcal{D}_1$	448	9

Table 1: The group 3-cubes of order $v = 16$.

Proposition 5.1. *Up to equivalence, the set $\mathcal{C}^3(16,6,2)$ contains exactly 27 difference cubes and 946 group cubes that are not difference cubes.*

The most interesting examples come from groups with IDs 5, 12, and 13, allowing difference sets for inequivalent designs. For example, a design over $\mathbb{Z}_8 \times \mathbb{Z}_2$ (group ID 5) isomorphic to $\mathcal{D}_3$ can be constructed so that 8 of its blocks are difference sets with development $\mathcal{D}_1$, and the other 8 blocks are difference sets with development $\mathcal{D}_2$. It gives rise to a

3-cube with slice invariant $\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^8, \mathcal{D}_2^8\}^1\}$ by Theorem 4.1. All slice invariants that occurred are given in Table 2, along with the corresponding numbers of difference cubes (Ndc) and non-difference group cubes (Ngc).

Slice invariant	Ndc	Ngc	Nng
$\{\{\mathcal{D}_1^{16}\}^3\}$	24	180	383
$\{\{\mathcal{D}_2^{16}\}^3\}$	1	20	32
$\{\{\mathcal{D}_3^{16}\}^3\}$	2	41	1
$\{\{\mathcal{D}_1^{16}\}^2, \{\mathcal{D}_2^{16}\}^1\}$	0	15	392
$\{\{\mathcal{D}_1^{16}\}^2, \{\mathcal{D}_3^{16}\}^1\}$	0	46	2
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_1^{16}\}^1\}$	0	284	444
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_3^{16}\}^1\}$	0	53	0
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^{16}\}^1\}$	0	189	77
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_2^{16}\}^1\}$	0	14	0
$\{\{\mathcal{D}_1^{16}\}^2, \{\mathcal{D}_1^8, \mathcal{D}_2^8\}^1\}$	0	6	72
$\{\{\mathcal{D}_1^{16}\}^2, \{\mathcal{D}_1^8, \mathcal{D}_3^8\}^1\}$	0	15	0
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_1^{12}, \mathcal{D}_2^4\}^1\}$	0	4	0
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_1^8, \mathcal{D}_2^8\}^1\}$	0	6	0
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_1^4, \mathcal{D}_2^{12}\}^1\}$	0	4	0
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_1^{12}, \mathcal{D}_3^4\}^1\}$	0	10	0
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_1^8, \mathcal{D}_3^8\}^1\}$	0	15	0
$\{\{\mathcal{D}_2^{16}\}^2, \{\mathcal{D}_1^4, \mathcal{D}_3^{12}\}^1\}$	0	10	0
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^{12}, \mathcal{D}_2^4\}^1\}$	0	2	0
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^8, \mathcal{D}_2^8\}^1\}$	0	4	0
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^4, \mathcal{D}_2^{12}\}^1\}$	0	2	0
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^{12}, \mathcal{D}_3^4\}^1\}$	0	6	0
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^8, \mathcal{D}_3^8\}^1\}$	0	14	0
$\{\{\mathcal{D}_3^{16}\}^2, \{\mathcal{D}_1^4, \mathcal{D}_3^{12}\}^1\}$	0	6	0
$\{\{\mathcal{D}_1^4, \mathcal{D}_2^{12}\}^3\}$	0	0	8
$\{\{\mathcal{D}_1^{12}, \mathcal{D}_2^4\}^2, \{\mathcal{D}_1^4, \mathcal{D}_2^{12}\}^1\}$	0	0	12

Table 2: Slice invariants of cubes in $\mathcal{C}^3(16, 6, 2)$.

We have also found cubes of symmetric designs not equivalent with any group cube, which we shall call *non-group cubes*. Using our package PAG [24], we computed the full autotopy groups of available cubes $C \in \mathcal{C}^3(16, 6, 2)$. We then chose subgroups $G \leq \mathrm{Atop}(C)$ and used the Kramer-Mesner method to construct all 3-cubes of $(16, 6, 2)$ designs with G as prescribed autotopy group. Clearly we must always get the cube C we started from, but often we also get other inequivalent cubes, some of which are non-group cubes. Here is an example.

Example 5.2. Let $G = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ be the group generated by the permutations

$$\begin{aligned}
\alpha_1 =& (1,16)(4,5)(6,11)(7,9)(8,10)(14,15)\\
& (17,28)(20,21)(22,27)(23,26)(24,25)(31,32)\\
& (33,44)(34,37)(35,36)(38,39)(40,41)(47,48),\\
\alpha_2 =& (1,14,2)(3,16,15)(4,13,6)(5,12,11)(8,9,10)\\
& (17,20,29)(18,27,32)(19,22,31)(21,30,28)(23,24,25)\\
& (33,47,46)(34,36,37)(38,40,42)(39,41,43)(44,48,45),\\
\alpha_3 =& (1,13)(2,11)(3,6)(7,8)(12,16)(14,15)\\
& (17,30,27,18)(19,28,29,22)(20,32,21,31)(23,25,24,26)\\
& (33,43,38,46)(34,36,35,37)(39,45,44,42)(40,48,41,47).
\end{aligned}$$

This is a group of order 384 isomorphic to $\mathbb{Z}_2^6 \rtimes S_3$ acting on the sets $\mathcal{P}_1 = \{1,\ldots,16\}$, $\mathcal{P}_2 = \{17,\ldots,32\}$, and $\mathcal{P}_3 = \{33,\ldots,48\}$. The sets are point classes of a transversal design (see page 5 for the definition) with blocks being G-orbits generated by the subsets

$$\{1,17,33\},\ \{1,17,40\},\ \{1,18,33\},\ \{1,18,34\},\ \{1,18,42\},$$
$$\{1,23,34\},\ \{1,23,40\},\ \{7,17,35\},\ \{7,17,40\},\ \{7,23,33\}.$$

This transversal design represents a 3-cube of $(16,6,2)$ designs with slice invariant $\{\{\mathcal{D}_1^4, \mathcal{D}_2^{12}\}^3\}$. It is not equivalent with any group cube, because parallel slices in all three directions comprise both designs $\mathcal{D}_1$ and $\mathcal{D}_2$. A 3-cube constructed from Theorem 4.1 must have slices isomorphic to $\mathcal{D}$ in two directions.

We found many non-group cubes of $(16,6,2)$ designs by this method.

Proposition 5.3. *The set* $\mathcal{C}^3(16,6,2)$ *contains at least* 1423 *inequivalent non-group cubes.*

The last column of Table 2 (Nng) contains their distribution by slice invariants. The constructed examples are available on our web page [23]. The total number of non-group cubes in $\mathcal{C}^3(16,6,2)$ is probably much larger. We have attempted construction only for groups of order $|G| \geq 512$.

We did not find any other non-group cubes for smaller parameters (v,k,λ), and for larger parameters the Kramer-Mesner approach was too inefficient. However, we did find larger non-difference group cubes by the construction of Theorem 4.1. A complete classification was possible for $(21,5,1)$. The non-difference group cube C_3 of Example 4.2 is unique.

Proposition 5.4. *The set* $\mathcal{C}^3(21,5,1)$ *contains exactly three inequivalent group cubes, two of which are difference cubes.*

For parameters $(27,13,6)$ we did an incomplete search and found 7 other group cubes besides the two difference cubes mentioned at the end of Section 3. For even larger parameters we searched for designs with difference sets as blocks and a prescribed automorphism group. The parameters for which we found non-difference group cubes are given in Table 3. An on-line version of the table with links to files containing the cubes is available on the web page [23]. The column Nds contains numbers of inequivalent difference sets according to [30]. For parameters $(64,28,12)$ some of the corresponding difference 3-cubes are equivalent, but we did not perform a complete enumeration.

Parameters	Nds	Ndc	Ngc
$(21, 5, 1)$	2	2	1
$(27, 13, 6)$	3	2	≥ 7
$(36, 15, 6)$	35	35	≥ 373
$(45, 12, 3)$	2	2	≥ 6
$(63, 31, 15)$	6	6	≥ 9
$(64, 28, 12)$	330159	< 330159	≥ 1
$(96, 20, 4)$	2627	1806	≥ 1

Table 3: Difference and group cubes for $n = 3$.

ORCID iDs

Vedran Krčadinac https://orcid.org/0000-0001-5199-0911
Mario Osvin Pavčević https://orcid.org/0000-0003-0433-2494
Kristijan Tabak https://orcid.org/0000-0002-7030-4945

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