Time-dependent equations governing the shape of a three-dimensional liquid curtain

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In a previous paper, Weinstein *et al.* derive time-dependent equations that govern the response of a planar liquid curtain falling under the influence of gravity and subjected to ambient pressure disturbances. In the previous study, disturbances to the curtain are assumed to be two-dimensional, and thus, the curtain response is independent of widthwise location in the curtain. In this paper, we generalize the previous equations to incorporate the widthwise dimension. The validity of these equations is demonstrated by their ability to predict standing wave shapes in agreement with those studied by Lin and Roberts. © *1998 American Institute of Physics.* [S1070-6631(98)03008-6]

I. INTRODUCTION

In a previous paper, Weinstein *et al.*[1] derive two equations governing the shape of a planar liquid sheet (a curtain) falling under the influence of gravity and subjected to ambient pressure disturbances. The two equations are termed sinuous and varicose, reflecting the types of curtain response described by each. In particular, the varicose equation governs thickness variations in the curtain, in which the two air–liquid interfaces move out of phase. The sinuous equation governs the deflection of the curtain centerline, and is characterized by interfaces that move in phase such that the local thickness of the curtain is preserved. While a general curtain disturbance will generally invoke both sinuous and varicose responses simultaneously, Weinstein *et al.* show that to a leading order approximation, pressure disturbances invoke a sinuous deflection with no varicose contribution. Weinstein *et al.* assume that the disturbances to the curtain are strictly two-dimensional, i.e., they are assumed to be independent of the widthwise direction z, oriented outward in Fig. 1. Under these two-dimensional circumstances, the curtain shape predictions of the sinuous governing equation have been verified experimentally for cases of a steady applied pressure[2] and time-varying pressure disturbances.[3]

In this paper, we incorporate the effects of the widthwise dimension by generalizing the previously derived equations of Weinstein *et al.*[1] Since the steady and time-dependent characters of the sinuous equation have already been verified experimentally as cited above, we choose to verify the added predictive ability of our generalized equations by considering a well-known steady problem that inherently involves the third dimension. Indeed, we demonstrate the ability of the generalized sinuous equation to predict standing wave shapes in agreement with the experimentally verified theory of Lin and Roberts.[4] We note here that Lin and Roberts, who generate standing waves by placing an obstacle in the curtain, focus only on the shape of the standing wave front observed in the curtain. They do *not* examine the distortion of the liquid curtain associated with these standing waves. An advantage of our derived sinuous governing equation is the ability to predict the shape of a liquid curtain having standing waves. To demonstrate this point, we provide an example in which curtain shapes having standing waves are generated via a point disturbance in pressure.

II. THEORY

We now consider the theoretical extension of the equations of Weinstein *et al.*[1] In their approach, a liquid curtain of constant density ρ and constant surface tension σ is examined as a potential flow, the geometry of which is shown in Fig. 1. The assumption that viscosity is negligible is justified by the large magnitude of the characteristic Reynolds number in the curtain, which is given by the reciprocal of the quantity in equation (2c) of Weinstein *et al.*[1] Their governing equations are derived under conditions in which the disturbances to the curtain are small, and thus, the disturbance equations are obtained via linearization about the undisturbed curtain flow. In the undisturbed case, the flow depends only on the x and y coordinates in Fig. 1 and is symmetrical about the line $y=0$; the ambient pressure exerted on the curtain is the atmospheric pressure, P_A. Due to the effects of gravity, the curtain thins, and the base flow is difficult to obtain exactly. However, an asymptotic solution of the potential flow system can be obtained in the limit of a long and thin curtain. This limit is characterized by small values of the aspect ratio parameter, ε, and $O(1)$ values of the Weber number, W, given by

$$\varepsilon=\frac{qg}{V^3},\quad W=\frac{\sigma}{\rho q V}. \tag{1}$$

In Eq. (1), q is the volumetric flow per unit width fed to the undisturbed curtain, g is the gravitational constant, and V is the average speed of the curtain at its top (i.e., $x=0$ in Fig. 1), which can thus be expressed in terms of the thickness there, d, as $V=q/d$. The leading order results for the undisturbed flow are the free-fall liquid speed, u, curtain thickness, h, and internal curtain pressure, P, given by

$$u=[V^2+2gx]^{1/2},\quad h=\frac{q}{u},\quad P=P_A. \tag{2}$$

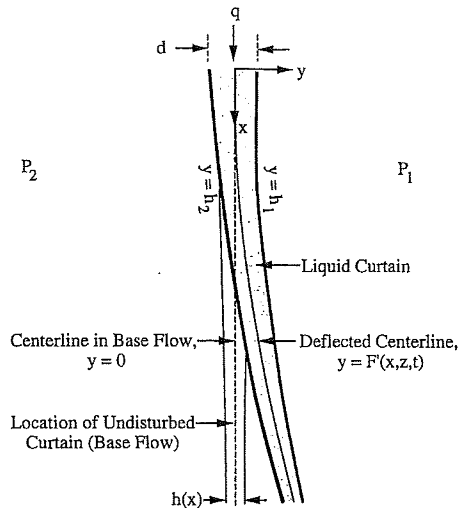

FIG. 1. Side view schematic of a curtain issuing from a slot and falling under the influence of gravity. The widthwise z-direction is oriented out of the figure.

Higher order corrections, which are invoked in powers of ε^2, reveal that the velocity profile is curved across the curtain cross section. These corrections to the base flow are relevant to the analysis of the *disturbed* curtain in the linearization process; Weinstein *et al.*[1] demonstrate that the approximate nature of the base flow requires the velocity field be known to $O(\varepsilon^2)$ so that the correct asymptotic ordering of the linearization can be achieved. In the current problem, the base flow is *identical* to that of Weinstein *et al.*[1] since the two-dimensional nature of the base flow inherently neglects the widthwise dimension z. The complete base flow solution is given by (13) in Weinstein *et al.*[1]

The system of equations governing the unsimplified two-dimensional time-dependent flow is given by (15) in Weinstein *et al.*[1] In the current problem, the z dependence is added and the modified system of governing equations is given by

$$\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2}+\varepsilon^2\left(\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2}+\frac{\partial^2\bar{\phi}}{\partial\bar{z}^2}\right)=0, \tag{3a}$$

$$\left(\frac{\partial\bar{\phi}}{\partial\bar{y}}\right)^2+\varepsilon^2\left[2\bar{P}+\left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\right)^2+\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)^2+2\frac{\partial\bar{\phi}}{\partial\bar{t}}-2\bar{x}-\bar{C}\right]=0, \tag{3b}$$

$$\frac{\partial\bar{\phi}}{\partial\bar{y}}=\varepsilon^2\left[\frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\partial\bar{h}_i}{\partial\bar{x}}+\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{h}_i}{\partial\bar{z}}+\frac{\partial\bar{h}_i}{\partial\bar{t}}\right]\quad \text{at } \bar{y}=\bar{h}_i(\bar{x},\bar{z},\bar{t})$$

$$\text{for } i=1,2, \tag{3c}$$

where, in anticipation of the $\varepsilon\rightarrow 0$ limit as used for the base flow, we can approximate the dynamic pressure constraint as

$$\bar{P}-\bar{P}_i\sim W\varepsilon^2(-1)^i\left(\frac{\partial^2\bar{h}_i}{\partial\bar{x}^2}+\frac{\partial^2\bar{h}_i}{\partial\bar{z}^2}\right)+O(\varepsilon^4)$$

$$\text{at } \bar{y}=\bar{h}_i(\bar{x},\bar{z},\bar{t})\quad \text{for } i=1,2. \tag{3d}$$

In Eqs. (3a)–(3d), t is time, $\phi=\phi(x,y,z,t)$ is the velocity potential, $P=P(x,y,z,t)$ is the internal curtain pressure, and $P_1=P_1(x,z,t)$ and $P_2=P_2(x,z,t)$ are the applied pressures on the front and back air-liquid interfaces of the curtain located, respectively, at $y=h_1$ and $y=h_2$ (Fig. 1). The constant $\bar{C}$ is the Bernoulli constant. This is chosen to be the same as in the undisturbed base flow in anticipation of the linearization to follow; the constant itself is given by (13d) in Weinstein *et al.*[1] In Eqs. (3a)–(3d), the overbars denote dimensionless variables, which are defined as

$$\bar{x}=\frac{xg}{V^2},\ \bar{y}=\frac{y}{d},\ \bar{z}=\frac{zg}{V^2},\ \bar{P}=\frac{P}{\rho V^2},\ \bar{\phi}=\frac{\phi g}{V^3},\ \bar{t}=\frac{tg}{V};$$

$$\bar{h}_i=\frac{h_i}{d},\ \bar{P}_i=\frac{P_i}{\rho V^2}\quad \text{for } i=1,2. \tag{3e}$$

With the time-dependent equations developed, we now proceed to linearize them about the base flow. The formal expression of the linearization is given by the expansions (18) in Weinstein *et al.*,[1] once the perturbed variables, which are denoted with a prime ($'$), are interpreted as being z dependent as indicated above. Substituting these expansions into the system (3) and equating terms of like order, the linearized system of equations is identical in form to (19) of Weinstein *et al.*,[1] once the term $(-1)^i W(\partial^2\bar{h}_i'/\partial\bar{z}^2)$ is added to their dynamic pressure constraint (19d). With this modification, the approach of Weinstein *et al.*[1] is followed to achieve the z-dependent governing equations. The perturbations to the front and back interface locations, denoted, respectively, by $\bar{h}_1'$ and $\bar{h}_2'$, can be expressed in terms of the sinuous deflection of the curtain centerline location (Fig. 1), $y=F'(x,z,t)$, and the varicose thickness variations about this centerline, $G'(x,z,t)$, as

$$\bar{F}'=\frac{F'}{d}=\frac{(\bar{h}_1'+\bar{h}_2')}{2},\quad \bar{G}'=\frac{G'}{d}=\frac{(\bar{h}_1'-\bar{h}_2')}{2}. \tag{4}$$

Using the definitions (4), the sinuous governing equation is

$$\frac{1}{\bar{u}}\frac{\partial^2\bar{F}'}{\partial\bar{t}^2}+2\frac{\partial^2\bar{F}'}{\partial\bar{x}\partial\bar{t}}+\frac{\partial}{\partial\bar{x}}\left[(\bar{u}-2W)\frac{\partial\bar{F}'}{\partial\bar{x}}\right]$$

$$-2W\frac{\partial^2\bar{F}'}{\partial\bar{z}^2}=\frac{(\bar{P}_2-\bar{P}_1)}{\varepsilon^2}, \tag{5a}$$

where $\bar{u}$ is the dimensionless free fall speed given using (1) as

$$\bar{u}=\frac{u}{V}=[1+2\bar{x}]^{1/2}. \tag{5b}$$

The net effect of our modified analysis is the added z-dependent derivative term on the left hand side of (5a) compared with the two-dimensional sinuous equation (25a) of Weinstein *et al.*[1] The varicose equation (25b) of Weinstein *et al.* is unaffected in form, and is written here for convenience.

$$\frac{\partial\bar{G}'}{\partial\bar{t}}+\frac{\partial(\bar{u}\bar{G}')}{\partial\bar{x}}=0. \tag{5c}$$

This concludes our derivation of the governing equations generalized to the widthwise z dimension, which are given by (5).

III. RESULTS: STANDING WAVES

To examine the physics contained in the added z-dimension now imbedded in the sinuous equation (5a), we consider the time-independent form of (5a), i.e.,

$$\frac{\partial}{\partial \bar{x}}\left[(\bar{u}-2W)\,\frac{\partial \bar{F}'}{\partial \bar{x}}\right]-2W\,\frac{\partial^2 \bar{F}'}{\partial \bar{z}^2}=\frac{(\bar{P}_2-\bar{P}_1)}{\varepsilon^2}. \tag{6}$$

From standard techniques of equation classification,[5] we find that the sinuous equation (6) is hyperbolic provided that $\bar{u}>2W$. According to the wave propagation results of Lin[6] and Lin and Roberts,[4] it is necessary that $\bar{u}>2W$ to be satisfied in order for standing waves to form. This constraint assures that the flow is supercritical, i.e., that the free-fall speed, $\bar{u}$, is greater than the speed of propagation of waves in the moving liquid frame given by $(2W\bar{u})^{1/2}$; thus, wave fronts can be held stationary against the oncoming fluid stream at some angle. This physics supports the interpretation of the variable $\bar{x}$ in (6) as a time-like coordinate such as found in typical hyperbolic systems. The hyperbolic nature of (6) for $\bar{u}>2W$ suggests that we identify characteristic curves in the $\bar{x}-\bar{z}$ domain along which the disturbance may affect the curtain shape. The characteristics are obtained from standard techniques[5] to yield

$$\bar{z}=\pm\tfrac{2}{3}(2W)^{1/2}(\bar{u}-2W)^{1/2}(\bar{u}+4W)+K, \tag{7}$$

where K is an arbitrary constant, and $\bar{u}$ is given by (5b). We note here that the characteristic curves described by (7) are identical to the equations associated with the standing wave shapes as described by Lin and Roberts[4] (for the case in which viscosity is neglected in their analysis). Thus, the curtain shapes predicted by Eq. (6) will be associated with observable standing waves, delineating the domains of influence of the imposed boundary conditions according to standard hyperbolic theory.[5]

To demonstrate the ability of (6) to predict curtain shapes, we now consider a model problem for the geometry shown in Fig. 2. As indicated, narrow edge guides confine the thin edges of the curtain at $\bar{z}=\pm\bar{D}/2$, where $\bar{D}$ is the dimensionless width of the curtain. Let us assume that these edge guides pin the curtain centerline such that it is undeflected, and further assume that the curtain is undisturbed at its top ($\bar{x}=0$); thus, we have the following boundary conditions:

$$\bar{F}'=0 \quad \text{at } \bar{z}=\pm\bar{D}/2; \quad \bar{F}'=\frac{\partial \bar{F}'}{\partial \bar{x}}=0 \qquad \text{at } \bar{x}=0. \tag{8}$$

Let us further assume that a point disturbance of pressure is applied to a curtain at the location $\bar{x}=\bar{x}_0$ and $\bar{z}=\bar{z}_0$ of the form

$$\bar{P}_2-\bar{P}_1=\gamma\varepsilon^2\,\delta(\bar{x}-\bar{x}_0)\,\delta(\bar{z}-\bar{z}_0), \tag{9}$$

where δ denotes the Dirac delta function of the indicated arguments, and γ is a constant that determines the strength of the disturbance. The system to solve is thus given by Eq. (6) after substitution of (9), and subject to the boundary conditions (8).

The complete solution of our problem generally requires numerical computation, owing to the nonconstant coefficient

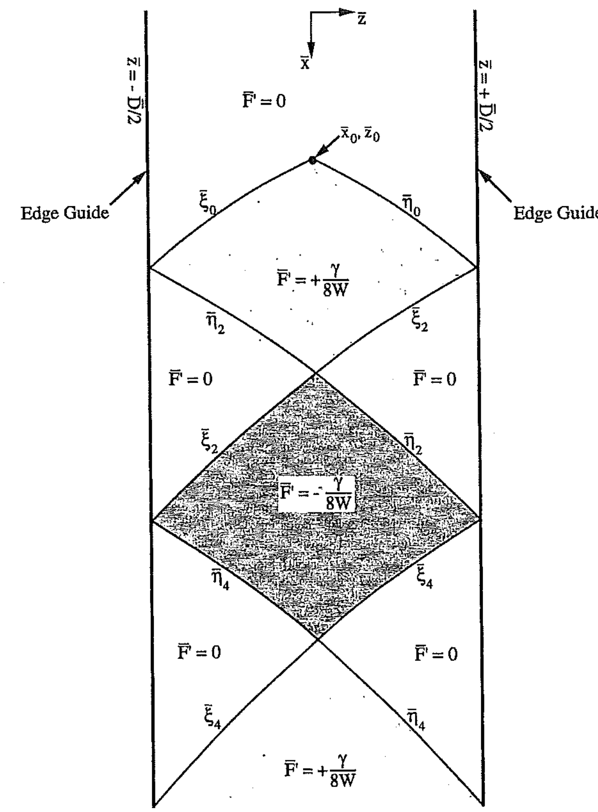

FIG. 2. Schematic view of the liquid curtain from its back air–liquid interface corresponding to $y=h_2$ in Fig. 1, showing the response of the curtain to a disturbance in pressure at the point x_0, z_0. The dimensionless curtain solution is given as $\bar{y}=\bar{F}'$, where the positive y direction is oriented into the figure.

nature of the governing equation (6). Nevertheless, we can obtain an approximate analytical solution as follows. First, we note that Eq. (7) can be used to define characteristic coordinates $\bar{\xi}$ and $\bar{\eta}$ as

$$\begin{aligned}\bar{\xi}&=\bar{z}+\tfrac{2}{3}(2W)^{1/2}(\bar{u}-2W)^{1/2}(\bar{u}+4W),\\ \bar{\eta}&=\bar{z}-\tfrac{2}{3}(2W)^{1/2}(\bar{u}-2W)^{1/2}(\bar{u}+4W),\end{aligned} \tag{10}$$

where for $\bar{\xi}=$constant or $\bar{\eta}=$constant, the standing wave equation (7) is obtained. We now transform Eq. (6) with Eq. (9) from $\bar{x}-\bar{z}$ to $\bar{\xi}-\bar{\eta}$ coordinates to obtain

$$\frac{\partial^2 \bar{F}}{\partial \bar{\xi}\partial \bar{\eta}}-\frac{2^{1/2}}{16W^{1/2}\bar{u}(\bar{u}-2W)^{1/2}}\left(\frac{\partial \bar{F}'}{\partial \bar{\xi}}-\frac{\partial \bar{F}'}{\partial \bar{\eta}}\right)=-\frac{\gamma}{8W}\,\delta(\bar{\xi}-\bar{\xi}_0)\,\delta(\bar{\eta}-\bar{\eta}_0). \tag{11}$$

In (11), $\bar{\xi}_0$ and $\bar{\eta}_0$ are the values of the characteristic curves passing through the disturbance location, obtained by substituting $\bar{u}_0$ and $\bar{z}_0$ into Eq. (10). In utilizing Eq. (11), the coefficients involving $\bar{u}$ are interpreted as being functions of $\bar{\xi}$ and $\bar{\eta}$. At this point, we note that because $\bar{u}$ increases with distance $\bar{x}$ down the curtain ($\bar{u}\geqslant 1>2W$), the coefficient of the bracketed term on the left-hand side may be small provided the disturbance location is far enough down the curtain. Under such circumstances, Eq. (11) becomes

$$\frac{\partial^2 \bar{F}'}{\partial \bar{\xi} \partial \bar{\eta}} \sim -\frac{\gamma}{8W} \delta(\bar{\xi}-\bar{\xi}_0)\delta(\bar{\eta}-\bar{\eta}_0). \tag{12}$$

Equation (12) can now be solved analytically, subject to boundary conditions (8) to determine the curtain response from standard techniques.

The solution of the problem is indicated in Fig. 2. The curtain is distorted into a series of diamond-shaped regions, which are found below the disturbance, in which the whole curtain is displaced forward or backward uniformly according to

$$\bar{F} = \pm \frac{\gamma}{8W}. \tag{13}$$

Outside of these regions, i.e., near the edge guides and top of the curtain, the interface is undistorted. Characteristic curves delineate the boundaries of the diamond-shaped regions as indicated in Fig. 2; these characteristic curves would be observed as standing waves in an experiment. The numerical values of the bounding characteristics are given recursively as

$$\bar{\xi}_{2n+2} = \bar{D} - \bar{\eta}_{2n}, \quad \bar{\eta}_{2n+2} = -\bar{D} - \bar{\xi}_{2n}, \quad n = 0,1,2,\ldots. \tag{14}$$

The equations for the curves corresponding to the constant values of the characteristics in (14), which are as indicated in Fig. 2, are obtained by substituting these constant values in the characteristic curve functions (10). Note that the discontinuous nature of our solution can be anticipated from standard hyperbolic theory;[5] in reality, the curtain shape is smooth across the discontinuities, but a local analysis would be required to resolve the precise curtain structure there. This completes the solution of our problem.

IV. SUMMARY AND CLOSING COMMENTS

In summary, then, we have demonstrated that the modified sinuous equation (5a), now extended to govern widthwise-varying shapes, can predict standing waves in liquid curtains. As suggested by Weinstein *et al.*,[1] a slightly more convenient form of the derived sinuous equation is obtained by utilizing a variable transformation between x and u according to the free-fall equation given in (2). The dimensional form of the transformed sinuous equation (5a) is

$$\left(\frac{\partial}{\partial t} + g\frac{\partial}{\partial u}\right)^2 F' - \frac{2\sigma g^2}{\rho q}\frac{\partial}{\partial u}\left[\frac{1}{u}\frac{\partial F'}{\partial u}\right] - \frac{2\sigma u}{\rho q}\frac{\partial^2 F'}{\partial z^2} = \frac{(P_2-P_1)u}{\rho q}. \tag{15}$$

We close this paper with a few comments regarding the steady state predictions of the varicose equation (5c). The general dimensional form of the steady state solution to Eq. (5c) is

$$G' = \frac{A(z)}{u}, \tag{16}$$

where $A(z)$ is an arbitrary function chosen depending upon appropriate boundary conditions. In the experiments of Lin and Roberts,[4] who examine relatively high viscosity fluids (>500 cP), varicose standing waves are observed that are roughly parabolic in shape; these are in addition to the sinuous waves predicted above. More recently, experiments by de Luca and Costa[7] have extended these varicose observations to low viscosity fluids (~1 cP), and clearly show that both sinuous and varicose standing waves can occur simultaneously. Our solution (16) does *not* admit such parabolic varicose waves; rather, the solution only allows for the prospect of thickness variations falling straight down in the curtain (such as streaks), i.e., the varicose "standing waves" have no breadth. Thus, our analysis predicts that sinuous and varicose standing waves (having breadth) cannot occur simultaneously, at least not to the order of the asymptotic approximations utilized. The simultaneous occurrence of varicose and sinuous standing waves observed in experiment likely indicates that a multiple scale analysis is required in the theory. Our asymptotic approximation does indeed remove the effects of surface tension in the varicose equation; such effects are required to achieve standing waves with breadth as can be seen from Lin and Roberts. Indeed, de Luca and Costa use a multiple scale approach in their examination of stationary waves[7] and curtain stability[8] in the Fourier domain. Future work may focus on augmenting the current analysis with a multiple scale approach.

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