Time-dependent equations governing the shape of a two-dimensional liquid curtain, Part 1: Theory

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(Received 5 February 1997; accepted 8 April 1997)

Approximate equations have been derived that govern the time-dependent response of a two-dimensional liquid curtain falling under the influence of gravity and subjected to ambient pressure disturbances. Starting with the assumptions of potential flow and constant surface tension, and using the approximation that the curtain is long and thin, a steady-state base flow is first determined. In agreement with previous literature results, the analysis reveals that the curtain flow is essentially in free fall, where the velocity profile is only slightly curved across the curtain thickness. Then, by assuming that the disturbances to the curtain are small, the time-dependent equations are linearized about the approximated base flow. The approximate nature of the base flow necessitates a careful ordering of terms to assure that the linearization is valid. Two equations governing the curtain shape are derived: the first governs the deflection of the curtain centerline, and the second governs the thickness variations. Previous literature results regarding wave propagation and steady curtain deflections can be predicted via the derived equations. It is also found that to lowest order, pressure disturbances induce a deflection of the curtain centerline while preserving the local thickness associated with the undisturbed curtain. [S1070-6631(97)02412-4]

I. INTRODUCTION

Two-dimensional planar liquid sheets, falling under the influence of gravity, are often employed to deposit uniform liquid layers on a solid moving substrate during coating processes. It is perhaps intuitive that a liquid sheet (henceforth called a curtain) is sensitive to disturbances in the ambient air that contacts its air–liquid interfaces with a significant surface area. Just as a shower curtain deflects inward toward the water stream emanating from a shower nozzle, a liquid curtain will deflect and respond to pressure disturbances in the surrounding air. The purpose of the current work is to provide a simplified set of time-dependent equations that govern the shape of a liquid curtain in response to ambient pressure disturbances. The simplifications are afforded by the natural length scales present in typical curtains (i.e., they are long and thin) and by the assumption that disturbances to the curtain are small.

To date, there has been little work done in this area, although some other related issues relevant to the current work have been addressed. We first focus on literature relevant to the *undisturbed* flow in the curtain itself. Brown[1] measured the velocity in a planar curtain issuing from a slot (having height D and width T, where $D \ll T$) and falling vertically under the influence of gravity with volumetric flow rate Q. Brown empirically obtained the following modified free-fall equation:

$$u=(V^2+2gx)^{1/2}, \tag{1a}$$

$$V^2=U^2-4\left(\frac{4\mu g}{\rho}\right)^{2/3}. \tag{1b}$$

In (1), u is the fluid velocity, $U=Q/(TD)$ is the average velocity at the slot exit, g is the gravitational constant, x is the vertical distance below the slot, μ is the liquid viscosity, and ρ is the density. Equation (1a) indicates that even when viscous effects are included, it is permissible to view the flow in the curtain as being in free fall with an adjusted starting velocity, V, at the lip. The modified starting velocity is primarily due to flow rearrangement in the curtain due to the loss in viscous traction as fluid issues from the slot itself. Equation (1) is not valid in the vicinity of the slot, as the velocity profile is clearly not flat during this flow rearrangement (for reference, we will henceforth call the flow rearrangement region the "entrance region"). Note that many theoretical studies have been performed to specifically examine the entrance region flow in detail;[2–6] these studies also indicate the importance of surface tension, which is not contained in the speed adjustment in Brown's formula shown in (1b).

In the Appendix of Brown,[1] Taylor derives the following theoretical equation governing the mean velocity, $u(x)$, in a curtain:

$$\frac{4\mu}{\rho}\frac{d}{dx}\left(\frac{1}{u}\frac{du}{dx}\right)-\frac{du}{dx}+\frac{g}{u}=0. \tag{2a}$$

The correctness of Taylor's equation away from the entrance region is verified by formal asymptotic methods in the derivation of Clarke[3] and more recently by Ramos.[7] We note here that an analytic solution of Eq. (2a) is provided by Clarke[3] in terms of Airy functions. Nevertheless, under conditions where the last two terms of (2a) are dominant over the first term, i.e., viscous effects are small, we obtain

$$\frac{du}{dx} \sim \frac{g}{u}, \tag{2b}$$

which upon integration leads to the simple free-fall result (1a). A formal condition to justify the neglect of the viscous term in (2a) is obtained by substituting (1a) into (2a) and taking the ratio of the viscous term to either of the remaining two, which yields

$$\frac{8g\mu}{\rho u^3} \ll 1, \tag{2c}$$

where u is given by (1a). Noting that the fluid accelerates in the liquid curtain with increasing x down the curtain, if (2b) is satisfied near the top of the curtain, fluid viscosity effects on the steady flow are small everywhere. Even in cases where the criterion (2c) is not satisfied near the top of a curtain, there will be locations farther downstream where the free-fall result will ultimately be a good approximation to the curtain flow, assuming the curtain is long enough. For cases where viscous effects are important, Brown[1] notes that for the range of conditions he surveyed in his experimental study, Eq. (1) is in close agreement with the solution of Taylor's equation. We note here that the quantity on the left-hand side of (2c) gives the relative magnitudes of viscous and inertial forces in the curtain, and as such, provides the relevant Reynolds number for the gravity-driven flow.

The free-fall equation (1a) has been used successfully in the modeling of curtain flows by Finnicum *et al.*,[8] who examine steady inviscid planar curtains subjected to an applied pressure drop via experimental and theoretical means. Additionally, experimental shapes of annular curtains and water bells (axisymmetric curtains) have been in good agreement with theory when a free-fall fluid speed is utilized. The reader is referred to Finnicum *et al.*[8] for a review of the relevant water-bell literature as it pertains to planar curtains. In summary, then, there is ample experimental and theoretical evidence indicating that, in most cases, the net effect of the flow rearrangement near the slot, as viewed downstream, is an apparent adjustment in the extrapolated average velocity of the fluid back to the slot [i.e., a modification of V in (1a)], and the downstream fluid flow can be viewed as being in free fall.

At this point, we observe that the free-fall result (1a) can be obtained from the inviscid Bernoulli equation for a case where constant atmospheric pressure is satisfied everywhere in the curtain's interior. Under such circumstances, the Bernoulli constant is the same everywhere in the liquid, suggesting that the flow is irrotational, and that the assumptions of potential flow are valid. However, when we assume that there exists a velocity potential, ϕ, which is defined in terms of the free-fall velocity u in the usual way, i.e., $u=\partial\phi/\partial x$, we find that the free-fall result does *not* satisfy the governing Laplace equation ($\nabla^2\phi=0$). Thus, the free-fall result (1a) cannot be exact from a theoretical point of view. Ramos[7] demonstrates that the free-fall result can be derived as the leading-order solution of the Navier–Stokes equations in the asymptotic limit of a "long and thin" curtain at large Reynolds numbers; higher-order corrections reveal that the velocity field is slightly curved across the curtain thickness. The analysis of Ramos focuses on flow regimes in which viscous effects are important, and the free-fall result he derives is a limiting case of the general viscous analysis. Using the same "long and thin" curtain approximation, de Luca and Costa[9] have recently derived the free-fall result, as well as the slightly curved velocity correction, starting from the assumption of inviscid flow. In this paper, we focus specifically on long and thin inviscid curtains as well. In utilizing potential flow assumptions to model the steady curtain flow downstream, it is implicit that the entrance region in the vicinity of the slot is responsible for dissipating the vorticity (the rotational aspect of the fluid imparted by the viscous drag along the solid walls) associated with the fluid in the slot.

We now consider the literature related to disturbances of the curtain itself. Lin[10] examines the propagation and stability of waves in a liquid curtain falling under the influence of gravity. His study is limited to long waves that travel one dimensionally (there is no dependence on the curtain width), and he identifies two distinct wave modes. The first is a sinuous wave, which when excited causes the front and back interfaces of the curtain to distort in sync; the thickness of the undisturbed base flow is thus preserved for a sinuous wave. The second mode is referred to as a varicose wave; here, the interfaces distort such that the local thickness of the curtain is disturbed. The sinuous waves will always propagate downward in the curtain provided that the local fluid velocity satisfies $W<\frac{1}{2}$ everywhere in the curtain, where W is a local Weber number defined as $W=\sigma/\rho q u$. Here, σ is the surface tension, ρ is the fluid density, q is the volumetric flow per width, and u is the local velocity in the curtain according to (1a). However, if $W>\frac{1}{2}$ for certain locations in the curtain, then waves can travel upward at those points. It is only such upward traveling sinuous waves that can grow; a condition of linear stability of the curtain is thus $W<\frac{1}{2}$ everywhere in the curtain. We note here that the fluid viscosity in the curtain does determine the growth rate, but not the speed of propagation, of these long waves. The nature of this curtain stability criterion is further examined via an inviscid analysis in the recent work of de Luca and Costa.[9] Note that Lin[10] and de Luca and Costa[9] focus on the stability of the liquid curtain and thereby examine individual Fourier modes for growth; they do not formulate the problem in a convenient way to determine the specific response to a given disturbance, which would generally involve the excitation of a number of such modes. Ramos[7] derives time-dependent equations for the shape of a viscous liquid curtain exposed to constant atmospheric pressure, and assumes the curtain shape is symmetrical about its vertical centerline. Thus, the analysis of Ramos is limited to varicose-type disturbances, and it cannot be used to predict curtain responses that invoke sinuous wave modes.

Finnicum *et al.*[8] consider the steady-state response of a planar curtain issuing from a slot and subjected to an applied constant pressure drop. They obtain theoretical predictions of interface shapes in excellent agreement with experiment. The curtain itself is assumed to be inviscid and locally plug; the thickness of the curtain slowly varies around the curtain centerline, which itself can be deflected due to the pressure drop. With these assumptions, a local macroscopic momentum balance is utilized to obtain a nonlinear equation governing the deflection of the curtain centerline due to the applied pressure. In mathematically modeling the curtain response, Finnicum *et al.* identify flow regimes in which the placement of the applied boundary conditions is different. In particular, for cases where $W<\frac{1}{2}$ everywhere in the curtain (recall $W=\sigma/\hat{\rho}qu$, where u is the local curtain speed), two boundary conditions for the curtain shape can be specified at the slot. The conditions are that the curtain centerline location and slope at the slot are specified. However, for cases where $W>\frac{1}{2}$ near the slot exit, the curtain velocity u will increase due to gravity, and the location $W=\frac{1}{2}$ may arise in the curtain before its bottom is reached. In such an event, the equations of Finnicum *et al.* have an apparent singularity at that location when a pressure drop is applied. Finnicum *et al.* show that the singularity is removable if the curtain adopts a shape having a particular slope at the location $W=\frac{1}{2}$. Thus, the two boundary conditions applied for this case are the location of the curtain at the slot exit, and the finite slope constraint downstream at the location where $W=\frac{1}{2}$. In both the $W<\frac{1}{2}$ or $W>\frac{1}{2}$ cases, Finnicum *et al.* observe that the locations of the applied boundary conditions are consistent with the direction of propagation of waves in the curtain; however, the relevance of this observation to the well-posedness of the theoretical curtain problem is not explored. Additionally, since the details of the local field equations are not considered in the macroscopic approach employed by Finnicum *et al.*, the rigorousness of the derived theoretical model is not obvious. These issues are resolved via analysis provided in this paper.

To summarize, then, the above-cited literature has not considered how the shape of a liquid curtain is affected by time-dependent pressure disturbances. In this paper, we derive a general set of governing equations suitable for determining the curtain response. These equations approximate the full set of governing equations by taking advantage of the facts that curtains are long and thin, and that the imposed pressure disturbances are small enough to allow linearization about a steady base flow. In Sec. II, the approximate solution for the steady undisturbed curtain flow is given; it is shown that the free-fall equation (1a) is an excellent asymptotic approximation to the fluid velocity profile, which is in reality curved across the curtain thickness, in agreement with the result of de Luca and Costa.[9] In Sec. III, we consider the time-dependent problem. In particular, in Sec. III A, the unsimplified time-dependent governing equations are provided. Since the base flow derived in Sec. II is approximate, a careful ordering of terms is required to assure that the linearization is valid; a determination of the required accuracy in the approximate base flow is provided in Sec. III B. In Sec. III C, we linearize the curtain response to pressure disturbances about the steady curtain solution derived in Sec. II to obtain the approximate governing equations for the curtain shape. The physical interpretation of the derived equations is provided in Sec. III D, and some useful alternative equation forms are provided. An assessment of the validity of the derived time-dependent equations is given in Sec. IV by comparing their predictions with those of the literature results cited above. In particular, it is demonstrated that the wave propagation results of Lin[10] are predicted by the derived equations, and the steady-state equation of Finnicum *et al.*[8] is the time-independent form of the derived equations. Furthermore, the mathematical justification for the location of the boundary conditions employed in Finnicum *et al.* is provided through examination of the time-dependent equations. A summary of the paper is provided in Sec. V. In part 2 of this paper, the predictive ability of the derived equations is demonstrated via a comparison with experiment.

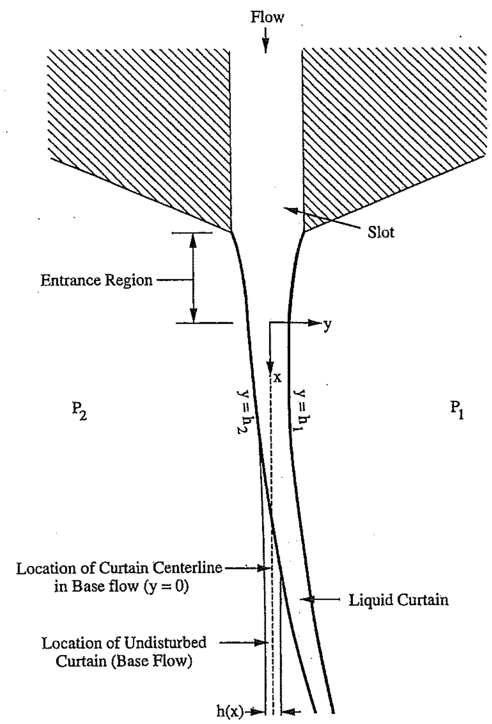

FIG. 1. Side view schematic of a curtain issuing from a slot and falling under the influence of gravity. The curtain is assumed to be infinite in the z direction, which is oriented out of the figure. The undisturbed base flow location of the curtain is also indicated in the figure, for which the interfaces are symmetrical about the line $y=0$, and the local curtain thickness is $h(x)$.

II. THE BASE CASE—UNDISTURBED FLOW IN THE CURTAIN

We begin by considering the case where there are no disturbances to the liquid curtain, which serves as the base case about which the time-dependent equations are linearized. To this end, consider the configuration shown in Fig. 1, where a planar liquid curtain emanates from a slot and accelerates due to gravity, g. The coordinate system is as shown in Fig. 1, where the x direction is oriented in the primary direction of flow, and the y direction is across the curtain thickness. Note that the coordinate system is located downstream of the entrance region; all analysis in this paper is focused on locations downstream of the region of flow rearrangement near the slot. We assume that the curtain is infinite in the widthwise z direction (oriented out of Fig. 1) and denote the external pressures exerted on the front and back faces of the curtain as P_1 and P_2, respectively; in the base flow, these pressures are equal to the constant atmospheric pressure, P_A. We also assume that the volumetric flow rate per width, q, is constant across the curtain width (in the z direction), and that both interfaces have constant surface tension σ. The locations of the front and back interfaces are denoted, respectively, as $y=h_1$ and $y=h_2$. In the undisturbed base flow, these interface locations can be expressed as $h_1=h(x)/2$ and $h_2=-h(x)/2$, where $h(x)$ is the local thickness of the curtain; the curtain is thus symmetrical about the centerline of the curtain, $y=0$, as shown in Fig. 1. The goal here is to determine the location-dependent velocity and pressure fields, as well as the local curtain thickness.

Motivated by the discussion in Sec. I, we assume that the liquid curtain can be modeled as a potential flow. To this end, we specify a velocity potential and pressure in the liquid, denoted by $\phi(x,y)$ and $P(x,y)$, respectively; the fluid velocity vector, $\mathbf{v}$, is related to the velocity potential as $\mathbf{v}=\nabla\phi$. We now choose relevant scales for the above-specified problem. We choose $y\sim d$ and $h(x)\sim d$, where d is the curtain thickness at the location $x=0$. The characteristic length scale in the x direction, L_s, is given by $L_s=V^2/g$, where V is the average curtain speed at $x=0$, and g is the gravitational constant. The volumetric flow rate per width delivered to the curtain is thus evaluated as $q=Vd$. The velocity potential and internal fluid pressure scale as $\phi\sim VL_s$ and $P\sim\rho V^2$, respectively. Using these scales, we define all dimensionless variables with an overbar as

$$\bar{x}=\frac{xg}{V^2},\quad \bar{y}=\frac{y}{d},\quad \bar{\phi}=\frac{\phi g}{V^3},\quad \bar{P}=\frac{P}{\rho V^2},\quad \bar{h}=\frac{h}{d},\tag{3a}$$

and the following dimensionless parameters naturally arise:

$$\epsilon=\frac{d}{L_s}=\frac{gq}{V^3},\quad W=\frac{\sigma}{\rho qV},\quad \bar{P}_A=\frac{P_A}{\rho V^2}.\tag{3b}$$

The governing equations are given by

$$\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2}+\epsilon^2\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2}=0,\tag{3c}$$

$$\left(\frac{\partial\bar{\phi}}{\partial\bar{y}}\right)^2+\epsilon^2\left[2\bar{P}+\left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\right)^2-2\bar{x}-\bar{C}\right]=0,\tag{3d}$$

where $\bar{C}$ is the dimensionless Bernoulli constant whose value is determined through the course of the analysis. The kinematic and dynamic boundary conditions are given, respectively, by

$$\frac{\partial\bar{\phi}}{\partial\bar{y}}=\frac{1}{2}\epsilon^2\frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{d\bar{h}}{d\bar{x}},\quad \text{at } \bar{y}=\frac{1}{2}\bar{h}(\bar{x}),\tag{3e}$$

and

$$\bar{P}-\bar{P}_A=-\tfrac{1}{2}W\epsilon^2\bar{\kappa},\quad \text{at } \bar{y}=\tfrac{1}{2}\bar{h}(\bar{x}),\tag{3f}$$

where $\bar{\kappa}$ is the dimensionless interfacial curvature defined as

$$\bar{\kappa}=\frac{d^2\bar{h}/d\bar{x}^2}{[1+\frac{1}{4}\epsilon^2(d\bar{h}/d\bar{x})^2]^{3/2}}.\tag{3g}$$

The volumetric flow constraint on the curtain is written in dimensionless form as

$$1=2\int_0^{\bar{h}/2}\frac{\partial\bar{\phi}}{\partial\bar{x}}\,d\bar{y}.\tag{3h}$$

The curtain is symmetrical about $\bar{y}=0$, and satisfies

$$\frac{\partial\bar{\phi}}{\partial\bar{y}}=\frac{\partial\bar{P}}{\partial\bar{y}}=0,\quad \text{at } \bar{y}=0.\tag{3i}$$

Finally, to be consistent with our assumed scale for the curtain thickness at $\bar{x}=0$, we require

$$\bar{h}=1,\quad \text{at } \bar{x}=0.\tag{3j}$$

The system (3) is well posed to solve for the flow field and curtain thickness. We note here that once obvious scaling differences are resolved, the above system (3) is identical to that obtained by de Luca and Costa[9] (once the density of the gas phase is neglected in their analysis), except for the explicit appearance of the Bernoulli constant, the thickness constraint (3j), and the volumetric flow constraint (3h).

At this point, we examine the order of magnitude of the dimensionless groups in (3b). In particular, we note that typical curtains are long and thin, and thus the aspect ratio parameter ϵ is small. Furthermore, the Weber number, W, is typically of order 1, based on stability considerations cited in Sec. I. Thus, we are led to consider the asymptotic simplifications to (3) in the limit as $\epsilon\rightarrow 0$, holding W fixed. We thus expand dependent variables in this limit as

$$\bar{\phi}\sim\bar{\phi}_0(\bar{x},\bar{y})+\epsilon^2\bar{\phi}_2(\bar{x},\bar{y})+O(\epsilon^4),\tag{4a}$$

$$\bar{P}\sim\bar{P}_0(\bar{x},\bar{y})+\epsilon^2\bar{P}_2(\bar{x},\bar{y})+O(\epsilon^4),\tag{4b}$$

$$\bar{h}\sim\bar{h}_0(\bar{x})+\epsilon^2\bar{h}_2(\bar{x})+O(\epsilon^4).\tag{4c}$$

We also find that the Bernoulli constant, $\bar{C}$, must be expanded in the parameter ϵ as

$$\bar{C}\sim\bar{C}_0+\epsilon^2\bar{C}_2+O(\epsilon^4);\tag{4d}$$

this expansion is required because the velocity field at $\bar{x}=0$ is adjusted through the course of the asymptotic analysis to maintain the curtain thickness specified by (3j). The expansion (4) is substituted into the system of equations (3) and terms of like order in ϵ are equated. The lowest-order problem to be solved is given by

$$\frac{\partial^2\bar{\phi}_0}{\partial\bar{y}^2}=0,\tag{5a}$$

$$\left(\frac{\partial\bar{\phi}_0}{\partial\bar{y}}\right)^2=0,\tag{5b}$$

$$\frac{\partial\bar{\phi}_0}{\partial\bar{y}}=0,\quad \bar{P}_0=\bar{P}_A,\quad \text{at } \bar{y}=\frac{1}{2}\bar{h}_0(\bar{x}),\tag{5c}$$

$$1=2\int_0^{\bar{h}_0/2}\frac{\partial\bar{\phi}_0}{\partial\bar{x}}\,d\bar{y},\tag{5d}$$

$$\frac{\partial \bar{\phi}_0}{\partial \bar{y}} = \frac{\partial \bar{P}_0}{\partial \bar{y}} = 0, \quad \text{at } \bar{y}=0, \tag{5e}$$

$$\bar{h}_0 = 1, \quad \text{at } \bar{x}=0. \tag{5f}$$

By inspection of the above system, the following lowest-order results are obtained:

$$\bar{\phi}_0 = \bar{\phi}_0(\bar{x}), \tag{6a}$$

$$\bar{h}_0 = \left(\frac{d\bar{\phi}_0}{d\bar{x}} \right)^{-1}, \tag{6b}$$

$$\bar{P}_0 = \bar{P}_A. \tag{6c}$$

In (6), the lowest-order velocity potential $\bar{\phi}_0(\bar{x})$ is still undetermined, although it has been simplified in that there is no dependence on the thickness variable $\bar{y}$.

The function $\bar{\phi}_0(\bar{x})$ is determined by expanding equation (3d) to $O(\epsilon^2)$ using (4) and (6a), to yield

$$\frac{d\bar{\phi}_0}{d\bar{x}} = [(\bar{C}_0 - 2\bar{P}_0) + 2\bar{x}]^{1/2}. \tag{7}$$

Substituting (7) into the lowest-order curtain thickness expression (6b), utilizing (6c), and applying the boundary condition (5f), we evaluate the lowest-order Bernoulli constant as

$$\bar{C}_0 = 1 + 2\bar{P}_A. \tag{8}$$

This immediately implies that

$$\frac{d\bar{\phi}_0}{d\bar{x}} = [1 + 2\bar{x}]^{1/2}. \tag{9a}$$

The result (9a) is the lowest-order velocity field; for future manipulations, we find it convenient to explicitly define this lowest-order result as

$$\bar{u} \equiv \frac{d\bar{\phi}_0}{d\bar{x}}. \tag{9b}$$

The lowest-order velocity potential is thus given by

$$\bar{\phi}_0 = \tfrac{1}{3}\bar{u}^3 + D, \tag{9c}$$

where D is an arbitrary constant of integration that has no effect on flow but is written here for completeness. The lowest-order thickness of the curtain is given by

$$\bar{h}_0 = \frac{1}{\bar{u}}. \tag{9d}$$

The results (9) constitute the completed solution to the lowest-order problem.

As will be shown in Sec. III B, the next-order correction to the velocity field and curtain thickness is required to examine the linearized time-dependent problem, and we thus consider the next-order correction here. Additionally, the generation of a next-order result serves to demonstrate the self-consistency of the asymptotic approach utilized in the examination of the base flow. The next-order set of equations [i.e., $O(\epsilon^2)$] to be solved is given by

$$\frac{\partial^2 \bar{\phi}_2}{\partial \bar{y}^2} = -\frac{d^2 \bar{\phi}_0}{\partial \bar{x}^2}, \tag{10a}$$

$$\frac{1}{2}\left(\frac{d\bar{\phi}_2}{d\bar{y}} \right)^2 + \bar{P}_2 + \frac{d\bar{\phi}_0}{d\bar{x}} \frac{d\bar{\phi}_2}{d\bar{x}} = \frac{1}{2} \bar{C}_2, \tag{10b}$$

$$\frac{\partial \bar{\phi}_2}{\partial \bar{y}} = \frac{1}{2} \frac{\partial \bar{\phi}_0}{\partial \bar{x}} \frac{d\bar{h}_0}{d\bar{x}}, \quad \text{at } \bar{y} = \frac{1}{2} \bar{h}_0(\bar{x}), \tag{10c}$$

$$\bar{P}_2 = -\frac{1}{2} W \frac{d^2 \bar{h}_0}{d\bar{x}^2}, \quad \text{at } \bar{y} = \frac{1}{2} \bar{h}_0(\bar{x}) \tag{10d}$$

$$\bar{h}_2 = -2\left(\frac{d\bar{\phi}_0}{d\bar{x}} \right)^{-1} \int_0^{\bar{h}_0/2} \frac{d\bar{\phi}_2}{d\bar{x}} \, d\bar{y}, \tag{10e}$$

$$\frac{\partial \bar{\phi}_2}{\partial \bar{y}} = \frac{\partial \bar{P}_2}{\partial \bar{y}} = 0, \quad \text{at } \bar{y}=0, \tag{10f}$$

$$\bar{h}_2 = 0, \quad \text{at } \bar{x}=0. \tag{10g}$$

We note here that to obtain (10b), it is necessary to retain terms in the expansion of (3d) to $O(\epsilon^4)$ [which indeed yields an expression solely involving the $O(\epsilon^2)$ independent variables].

We now proceed to solve the system (10); to this end, we shall find it useful to evaluate derivatives of $\bar{\phi}_0$ and $\bar{h}_0$ that explicitly appear in (10) as

$$\frac{d^2\bar{\phi}_0}{d\bar{x}^2} = \frac{1}{\bar{u}}, \quad \frac{d\bar{h}_0}{d\bar{x}} = -\frac{1}{\bar{u}^3}, \quad \frac{d^2\bar{h}_0}{d\bar{x}^2} = \frac{3}{\bar{u}^5}, \tag{11}$$

where $\bar{u}$ and $\bar{h}_0$ are given in (9). Integrating (10a) yields

$$\bar{\phi}_2 = -\frac{1}{2\bar{u}} \bar{y}^2 + \bar{A}(\bar{x})\bar{y} + \bar{B}(\bar{x}), \tag{12a}$$

where $\bar{A}(\bar{x})$ and $\bar{B}(\bar{x})$ are arbitrary functions arising upon integration. Applying (10c) and the boundary condition on the velocity potential in (10f), we obtain

$$\bar{A}(\bar{x}) = 0. \tag{12b}$$

Combining (10b) and (10d), and utilizing (11), we obtain

$$\frac{1}{2}\left(\frac{d\bar{\phi}_2}{d\bar{y}} \right)^2 - \frac{3W}{2\bar{u}^5} + \bar{u}\frac{d\bar{\phi}_2}{d\bar{x}} = \frac{1}{2}\bar{C}_2, \quad \text{at } \bar{y} = \frac{1}{2}\bar{h}_0(\bar{x}). \tag{12c}$$

Substituting (12a) into (12c) yields

$$\frac{d\bar{B}}{d\bar{x}} = \frac{3W}{2\bar{u}^6} - \frac{1}{4\bar{u}^5} + \frac{\bar{C}_2}{2\bar{u}}, \tag{12d}$$

which, upon integration, gives

$$\bar{B} = \frac{1}{12\bar{u}^3} - \frac{3W}{8\bar{u}^4} + \frac{\bar{C}_2}{2}\bar{u}. \tag{12e}$$

In writing (12e), we have neglected the integration constant that arises without loss of generality; this is because we have already absorbed an arbitrary integration constant into the lowest-order potential (9c) [an arbitrary constant arising in (12e) would merely add to the lowest-order constant to yield a new arbitrary constant]. Although $\bar{C}_2$ is as yet undetermined, the velocity potential is sufficiently known to completely determine the pressure correction; this is obtained by substituting (12a), (12b), and (12e) into (10b) to yield

$$\bar{P}_2 = -\frac{1}{\bar{u}^2}\bar{y}^2 + \left(\frac{1}{4\bar{u}^4} - \frac{3W}{2\bar{u}^5}\right). \tag{12f}$$

To determine $\bar{C}_2$, we substitute (12a), and (12b), and (12c) into the volumetric flow constraint (10e), and employ (9d), to obtain

$$\bar{h}_2 = \frac{5}{24\bar{u}^7} - \frac{3W}{2\bar{u}^8} - \frac{\bar{C}_2}{2\bar{u}^3}. \tag{12g}$$

Application of the boundary condition (10g) to (12g) yields

$$\bar{C}_2 = \tfrac{5}{12} - 3W. \tag{12h}$$

This completely determines the $O(\epsilon^2)$ corrections to the curtain shape, pressure, and velocity fields.

The velocity potential, pressure, and interface solutions valid to $O(\epsilon^2)$ are obtained by substituting (6c), (9), and (12) into the expansions (4) to yield

$$\bar{\phi} \sim \frac{1}{3}\bar{u}^3 + D + \epsilon^2\left(-\frac{1}{2\bar{u}}\bar{y}^2 + \frac{1}{12\bar{u}^3} - \frac{3W}{8\bar{u}^4} + \frac{(5-36W)}{24}\bar{u}\right) + O(\epsilon^4), \tag{13a}$$

$$\bar{P} \sim \bar{P}_A + \epsilon^2\left[-\frac{1}{\bar{u}^2}\bar{y}^2 + \left(\frac{1}{4\bar{u}^4} - \frac{3W}{2\bar{u}^5}\right)\right] + O(\epsilon^4), \tag{13b}$$

$$\bar{h} \sim \frac{1}{u} + \epsilon^2\left(\frac{5}{24\bar{u}^7} - \frac{3W}{2\bar{u}^8} - \frac{(5-36W)}{24\bar{u}^3}\right) + O(\epsilon^4). \tag{13c}$$

The Bernoulli constant is given by substituting (8) and (12h) into (4d) to obtain

$$\bar{C} \sim (1+2\bar{P}_A) + \epsilon^2(\tfrac{5}{12} - 3W) + O(\epsilon^4). \tag{13d}$$

Using (13a), the fluid velocities in the x and y directions, denoted, respectively, as $\bar{v}_x$ and $\bar{v}_y$, are given by

$$\bar{v}_x = \frac{\partial\bar{\phi}}{\partial\bar{x}} \sim \bar{u} + \epsilon^2\left(\frac{1}{2\bar{u}^3}\bar{y}^2 + \frac{3W}{2\bar{u}^6} - \frac{1}{4\bar{u}^5} + \frac{(5-36W)}{24\bar{u}}\right) + O(\epsilon^4) \tag{13e}$$

$$\bar{v}_y = \frac{\partial\bar{\phi}}{\partial\bar{y}} \sim -\epsilon^2\left(\frac{\bar{y}}{\bar{u}}\right) + O(\epsilon^4), \tag{13f}$$

where in (13), $\bar{u}$ is given by (9a) and (9b) and D is an arbitrary constant. Our results (13) are identical to those of de Luca and Costa,[9] except for the appearance of terms involving the quantity (5-36W) in (13a), (13c), and (13e). These additional terms arise as a direct result of imposing the thickness constraint (3h), and adjusting the Bernoulli constant for the nonflat velocity field. Note that for large $\bar{u}$, these additional terms are important in that they govern the magnitude of the $O(\epsilon^2)$ correction.

Before leaving this section, we make a few comments about the steady flow results (13). In dimensional form, the $O(1)$ velocity, pressure, and thickness results are given by

$$v_x \sim u = (V^2 + 2gx)^{1/2}, \tag{14a}$$

$$h \sim \frac{q}{u}, \tag{14b}$$

$$P \sim P_A. \tag{14c}$$

By inspection, it is seen that the $O(1)$ results in (14) describe a fluid in free fall, where the curtain itself experiences atmospheric pressure everywhere. It is also apparent from (13) that for a given aspect ratio ϵ, higher-order corrections will become increasingly small with increasing x as the fluid accelerates (i.e., $\bar{u}$ increases). Thus, even in situations in which ϵ is small but non-negligible, the fluid speed itself may dictate that higher-order corrections are exceedingly small.

III. THE TIME-DEPENDENT PROBLEM

A. The unsimplified time-dependent governing equations

The geometry and coordinate system for the time-dependent problem are identical to those for the steady base flow discussed in Sec. II, as shown in Fig. 1. As in the steady problem, we will assume that both interfaces have constant surface tension, σ, and the fluid density, ρ, is constant. We restrict attention to pressure disturbances that are two dimensional and do not depend upon the curtain width. Thus, we assume that the imposed air pressure on the front and back faces of the curtain can be parametrized, respectively, as $P_1(x,t)$ and $P_2(x,t)$, where t is time; the respective locations of the front and back faces of the curtain are parametrized as $y=h_1(x,t)$ and $y=h_2(x,t)$. As in the steady base flow, we assume that the liquid curtain can be modeled as a potential flow. The time-dependent velocity potential and pressures in the liquid are denoted as $\phi(x,y,t)$ and $P(x,y,t)$, respectively. The primary goal here is to determine the velocity field and interface location as a function of location and time.

In anticipation of the linearization procedure to be used, we make variables dimensionless based on scales derived from the base flow. Dimensionless variables that appear in the base flow, which necessarily appear in the time-dependent flow, are given in (3a). Additional dimensionless variables that arise in the time-dependent problem are defined as

$$\bar{t} = \frac{tg}{V}; \quad \bar{h}_i = \frac{h_i}{d}, \quad \text{for } i=1,2. \tag{15a}$$

In (15a), d and V are defined as in Sec. II, i.e., they are the curtain thickness and average speed associated with the undisturbed base flow at $x=0$. The dimensionless equations governing the time-dependent potential flow are given by

$$\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} + \epsilon^2\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} = 0, \tag{15b}$$

$$\left(\frac{\partial\bar{\phi}}{\partial\bar{y}}\right)^2 + \epsilon^2\left[2\bar{P} + \left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\right)^2 + 2\frac{\partial\bar{\phi}}{\partial\bar{t}} - 2\bar{x} - \bar{C}(\bar{t})\right] = 0. \tag{15c}$$

The kinematic boundary conditions at the air–liquid interfaces are

$$\frac{\partial\bar{\phi}}{\partial\bar{y}}=\epsilon^2\left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\partial\bar{h}_i}{\partial\bar{x}}+\frac{\partial\bar{h}_i}{\partial\bar{t}}\right),\quad \text{at } \bar{y}=\bar{h}_i(\bar{x},\bar{t}),\quad \text{for } i=1,2. \tag{15d}$$

The dynamic boundary conditions at these locations are

$$\bar{P}-\bar{P}_i=(-1)^i We^2\bar{\kappa}_i,\quad \text{at } \bar{y}=\bar{h}_i(\bar{x},\bar{t}),\quad \text{for } i=1,2, \tag{15e}$$

where $\bar{\kappa}_i$ is the dimensionless interfacial curvature defined as

$$\bar{\kappa}_i=\frac{d^2\bar{h}_i/d\bar{x}^2}{[1+\epsilon^2(d\bar{h}_i/d\bar{x})^2]^{3/2}}. \tag{15f}$$

All dimensionless groups in (15) are identical to those in the steady problem, as defined in (3b). Additionally, in anticipation of the linearization to follow, we will choose the time-dependent Bernoulli function $\bar{C}(\bar{t})$ in (15c) to be identical to the constant value obtained in the steady flow, as given by (13d).

At this point, we acknowledge that additional boundary conditions are needed to make the above-specified system well posed. For example, details of the interface position at a specified location, the velocity field there, and the precise form of the pressure disturbances, are needed to close the problem. However, the system (15) is sufficient to allow for the determination of simplified governing equations for the interface shape and velocity field through linearization. The specification of these additional boundary conditions become apparent based on the structure of the derived governing equations and the physics of the particular problem examined.

B. Determination of the required accuracy of the approximate base flow to assure a valid time-dependent linearized solution

We now consider the linearization of the system (15) about the base flow solution (13). In Sec. II, we took advantage of the fact that the parameter ϵ was small to simplify the base flow analysis and, as a result, the base flow (13) is approximate and valid up to and including $O(\epsilon^2)$. When performing a linearization about an *approximate* base flow solution, we need to assure that the ordering of terms in the linearization is correct; that is, we need to assure that the linearization is consistent with the magnitude of terms neglected in the approximation of the base flow. This is in contrast to the case where the base flow is known *exactly* (for example, rectilinear flow down an inclined plane), for which the linearization procedure is straightforward and proceeds in the usual way.

With the above comments in mind, let us inspect the system (15) and deduce how the ordering of terms should proceed when considering a linearization. Let us assume that to accomplish a linearization, the velocity potential, pressure, and interface locations are perturbed as follows:

$$\bar{\phi}\sim\bar{\phi}(\bar{x},\bar{y})_{\rm BF}+\bar{\Phi}'(\bar{x},\bar{t}), \tag{16a}$$

$$\bar{P}\sim\bar{P}(\bar{x},\bar{y})_{\rm BF}+\bar{p}'(\bar{x},\bar{t}), \tag{16b}$$

$$\bar{h}_i\sim(-1)^{i+1}\frac{\bar{h}(\bar{x})}{2}+\bar{h}_i'(\bar{x},\bar{t}),\quad i=1,2. \tag{16c}$$

In (16), $\bar{h}(\bar{x})$ is the base flow curtain thickness given by (13c) and the base flow velocity potential and pressure, denoted with the subscript "BF," are given by (13a) and (13b). Additionally, the primed quantities are assumed to be small relative to the base flow variables. Inspection of (13) indicates that all the base flow variables are approximated as an asymptotic series in the small parameter ϵ^2; at this point, it is not clear how many terms in the base flow expansion (13) are sufficient to specify the base flow, such that the linearization is valid (note that the parameter ϵ must be assumed small in the linearization to be consistent with the derived base flow). To resolve this issue, we focus attention on the boundary condition (15d) applied at the front interface ($i=1$), which relates its location to the velocity potential. Substituting (16) into (15d), neglecting terms that are quadratic or higher in the perturbed quantities, and equating terms of like order, we obtain

$$\frac{\partial\bar{\phi}_{\rm BF}}{\partial\bar{y}}=\frac{1}{2}\epsilon^2\frac{\partial\bar{\phi}_{\rm BF}}{\partial\bar{x}}\frac{d\bar{h}}{d\bar{x}},\quad \text{at } \bar{y}\sim\frac{1}{2}\bar{h}(\bar{x}), \tag{17a}$$

$$\frac{\partial\bar{\Phi}'}{\partial\bar{y}}=\epsilon^2\left(\frac{\partial\bar{\phi}_{\rm BF}}{\partial\bar{x}}\frac{\partial\bar{h}_1'}{\partial\bar{x}}+\frac{1}{2}\frac{\partial\bar{\Phi}'}{\partial\bar{x}}\frac{d\bar{h}}{d\bar{x}}+\frac{\partial\bar{h}_1'}{\partial\bar{t}}\right),$$
$$\text{at } \bar{y}\sim\tfrac{1}{2}\bar{h}(\bar{x}). \tag{17b}$$

In writing (17), note that we have not yet employed the $\epsilon\to 0$ limit. As expected in a self-consistent linearization, Eq. (17a) is identical to (3e), which is satisfied by the base flow. Turning attention to (17b), which relates the perturbed quantities, we note that perturbations in the interface location interact with a perturbed velocity potential; inspection of (17b) indicates that $\partial\bar{\Phi}'/\partial\bar{y}$ is of $O(\epsilon^2)$ for a given perturbation $\bar{h}_1'$. Examination of the other boundary condition (15e) in the same way confirms that $O(\epsilon^2)$ perturbations in the pressure interact with $O(1)$ perturbations in the interface location $\bar{h}_1'$; the same structure is also found for the back interface $\bar{h}_2'$. These arguments suggest that $\bar{\Phi}'$ and $\bar{p}'$ must be of $O(\epsilon^2)$ in our linearization while $\bar{h}_1'$ and $\bar{h}_2'$ are treated as perturbations at $O(1)$. The above arguments indicate that in linearizing about the base flow, it is necessary to know the pressure and the velocity potential in the base flow, given by (13a) and (13b), up to and including $O(\epsilon^2)$, while it is only necessary to know the base flow interface location (13c) to $O(1)$. To this end we explicitly express $\bar{\Phi}'$ and $\bar{p}'$ in (16) as

$$\bar{\Phi}'=\epsilon^2\bar{\phi}' \tag{17c}$$

$$\bar{p}'=\epsilon^2\bar{P}', \tag{17d}$$

where $\bar{\phi}'$ and $\bar{P}'$ are the correctly ordered linearized functions.

C. The linearized time-dependent equations

Based on the discussion in Sec. III B, the formal expression of the linearization of (15) is obtained by explicitly rewriting the expansion (16) using (17c) and (17d) and substituting the base flow results (13) to obtain

$$\bar{\phi}\sim\left[\frac{1}{3}\bar{u}^3+D+\epsilon^2\left(-\frac{1}{2\bar{u}}\bar{y}^2+\frac{1}{12\bar{u}^3}-\frac{3W}{8\bar{u}^4}+\frac{(5-36W)}{24}\bar{u}\right)+O(\epsilon^4)\right]+\epsilon^2[\bar{\phi}'(\bar{x},\bar{t})+O(\bar{\phi}'^2)], \tag{18a}$$

$$\bar{P}\sim\left\{\bar{P}_A+\epsilon^2\left[-\frac{1}{\bar{u}^2}\bar{y}^2+\left(\frac{1}{4\bar{u}^4}-\frac{3W}{2\bar{u}^5}\right)\right]+O(\epsilon^4)\right\}+\epsilon^2[\bar{P}'(\bar{x},\bar{t})+O(\bar{P}'^2)], \tag{18b}$$

$$\bar{h}_i\sim\left(\frac{(-1)^{i+1}}{2\bar{u}}+O(\epsilon^2)\right)+\bar{h}_i'(\bar{x},\bar{t})+O(\bar{h}_i'^2),\quad i=1,2. \tag{18c}$$

To be consistent with the form of the pressure expansion (18b) and after inspection of (15e), we assume that the imposed pressure fluctuations along the front and back faces of the curtain are given by

$$\bar{P}_i=\bar{P}_A+\epsilon^2\bar{P}_i'(\bar{x},\bar{t}),\quad i=1,2. \tag{18d}$$

Note that the form of (18d) assures that the response of the curtain to the disturbance is consistent with the linear assumption. Then, substituting (18) into the system (15) and equating terms of like order, the linear system governing the time-dependent flow is given by

$$\frac{\partial^2\bar{\phi}'}{\partial\bar{y}^2}=0, \tag{19a}$$

$$\bar{P}'=-\bar{u}\frac{\partial\bar{\phi}'}{\partial\bar{x}}-\frac{\partial\bar{\phi}'}{\partial\bar{t}}+\frac{\bar{y}}{\bar{u}}\frac{\partial\bar{\phi}'}{\partial\bar{y}}, \tag{19b}$$

$$\frac{\partial\bar{\phi}'}{\partial\bar{y}}=\frac{\partial\bar{h}_i'}{\partial\bar{t}}+\frac{\partial}{\partial\bar{x}}(\bar{u}\bar{h}_i'),\quad \text{at } \bar{y}=\frac{(-1)^{i+1}}{2\bar{u}},\quad i=1,2, \tag{19c}$$

$$\bar{P}'-\bar{P}_i'=(-1)^i\left(W\frac{\partial^2\bar{h}_i'}{\partial\bar{x}^2}-\frac{\bar{h}_i'}{\bar{u}^3}\right),\quad \text{at } \bar{y}=\frac{(-1)^{i+1}}{2\bar{u}},\quad i=1,2. \tag{19d}$$

We now solve the system (19) to obtain simplified equations governing the interface locations and flow. Integrating (19a), we obtain

$$\bar{\phi}'=\bar{A}'(\bar{x},\bar{t})\bar{y}+\bar{B}'(\bar{x},\bar{t}), \tag{20a}$$

where $\bar{A}'$ and $\bar{B}'$ are as of yet undetermined functions arising due to integration, which are evaluated upon application of the boundary conditions (19c) and (19d). Substitution of (20a) in (19c) yields the following:

$$\bar{A}'=\frac{\partial\bar{h}_1'}{\partial\bar{t}}+\frac{\partial}{\partial\bar{x}}(\bar{u}\bar{h}_1'), \tag{20b}$$

$$\bar{A}'=\frac{\partial\bar{h}_2'}{\partial\bar{t}}+\frac{\partial}{\partial\bar{x}}(\bar{u}\bar{h}_2'). \tag{20c}$$

Adding (20b) and (20c), we obtain

$$\bar{A}'=\frac{1}{2}\frac{\partial}{\partial\bar{t}}(\bar{h}_1'+\bar{h}_2')+\frac{1}{2}\frac{\partial}{\partial\bar{x}}[\bar{u}(\bar{h}_1'+\bar{h}_2')], \tag{21a}$$

while subtracting the same equations yields

$$\frac{\partial}{\partial\bar{t}}(\bar{h}_1'-\bar{h}_2')+\frac{\partial}{\partial\bar{x}}[\bar{u}(\bar{h}_1'-\bar{h}_2')]=0. \tag{21b}$$

Substituting the expression for $\bar{P}'$ given by (19b) into (19d) and substituting the velocity potential (20a) into the result, we obtain

$$\frac{1}{2}\left(\frac{\partial}{\partial\bar{x}}+\frac{1}{\bar{u}}\frac{\partial}{\partial\bar{t}}-\frac{1}{\bar{u}^2}\right)\bar{A}'+\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}\right)\bar{B}'+\frac{\bar{h}_1'}{\bar{u}^3}+\bar{P}_1'=W\frac{\partial^2\bar{h}_1'}{\partial\bar{x}^2}, \tag{22a}$$

$$\frac{1}{2}\left(\frac{\partial}{\partial\bar{x}}+\frac{1}{\bar{u}}\frac{\partial}{\partial\bar{t}}-\frac{1}{\bar{u}^2}\right)\bar{A}'-\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}\right)\bar{B}'+\frac{\bar{h}_2'}{\bar{u}^3}-\bar{P}_2'=W\frac{\partial^2\bar{h}_2'}{\partial\bar{x}^2}. \tag{22b}$$

Adding (22a) and (22b) yields

$$\left(\frac{\partial}{\partial\bar{x}}+\frac{1}{\bar{u}}\frac{\partial}{\partial\bar{t}}-\frac{1}{\bar{u}^2}\right)\bar{A}'+\frac{(\bar{h}_1'+\bar{h}_2')}{\bar{u}^3}+(\bar{P}_1'-\bar{P}_2')=W\frac{\partial^2}{\partial\bar{x}^2}(\bar{h}_1'+\bar{h}_2'), \tag{23a}$$

while subtracting the same equations yields

$$\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}\right)\bar{B}'+\frac{(\bar{h}_1'-\bar{h}_2')}{2\bar{u}^3}+\frac{1}{2}(\bar{P}_1'+\bar{P}_2')=\frac{W}{2}\frac{\partial^2}{\partial\bar{x}^2}(\bar{h}_1'-\bar{h}_2'). \tag{23b}$$

At this point, we comment that (21) and (23), once combined, yield simplified equations governing the interface locations and the perturbed flow field. Before writing these desired expressions, we note that terms involving the unknown interface locations appear in quantities of the form $(\bar{h}_1'+\bar{h}_2')$ and $(\bar{h}_1'-\bar{h}_2')$. For compactness, let us define the following quantities:

$$\bar{F}'=\frac{(\bar{h}_1'+\bar{h}_2')}{2}, \tag{24a}$$

$$\bar{G}'=\frac{(\bar{h}_1'-\bar{h}_2')}{2}. \tag{24b}$$

Substituting (21a) into (23a), employing (24a), and simplifying, we obtain

$$\frac{1}{\bar{u}}\frac{\partial^2\bar{F}'}{\partial\bar{t}^2}+2\frac{\partial^2\bar{F}'}{\partial\bar{x}\,\partial\bar{t}}+\frac{\partial}{\partial\bar{x}}\left((\bar{u}-2W)\frac{\partial\bar{F}'}{\partial\bar{x}}\right)=(\bar{P}_2'-\bar{P}_1'). \tag{25a}$$

An equation governing $\bar{G}'$ can be written directly by substituting (24b) into (21b):

$$\frac{\partial\bar{G}'}{\partial\bar{t}}+\frac{\partial(\bar{u}\bar{G}')}{\partial\bar{x}}=0. \tag{25b}$$

The functions $\bar{A}'$ and $\bar{B}'$ in the perturbed velocity potential (20a) are given in terms of $\bar{F}'$ and $\bar{G}'$ as follows. By substituting (24a) into (21a), we obtain

$$\bar{A}'=\frac{\partial\bar{F}'}{\partial\bar{t}}+\frac{\partial(\bar{u}\bar{F}')}{\partial\bar{x}}. \tag{25c}$$

Substituting (24b) into (23b), an equation to solve for $\bar{B}'$ is given by

$$\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}\right)\bar{B}'=W\frac{\partial^2\bar{G}'}{\partial\bar{x}^2}-\frac{\bar{G}'}{\bar{u}^3}-\frac{1}{2}(\bar{P}_1'+\bar{P}_2'). \tag{25d}$$

The linearized system of equations (25) can be solved to obtain the perturbed interface locations $\bar{h}_1'$ and $\bar{h}_2'$ [which are extracted from from $\bar{F}'$ and $\bar{G}'$ according to (24)], as well as the perturbed velocity potential (20a) (through the solution of $\bar{A}'$ and $\bar{B}'$). The system (25) is the desired result of our derivation. We provide a physical interpretation of the derived equations in the next section, and an assessment of their validity in Sec. IV.

D. Interpretation of derived equations, and alternative forms

We now discuss the physical implications of the derived equations (25), reverting to dimensional form for this purpose. The functions F' and G' in (25a) and (25b) correspond to different types of perturbations to the curtain shape. We interpret the disturbance function F' in (24a), which is the average location of the front and back interfaces, as yielding the deflection of the centerline of the curtain due to a disturbance (Fig. 2); the centerline deflection is a purely sinuous response, where the front and back interfaces move in sync. Then, as indicated in Fig. 2, the free-fall local curtain thickness h, given by (14b), is preserved, even during a centerline deflection. Such an interpretation is supported by the asymptotic expansion (18) and a discussion in Sec. III B, where higher-order corrections to the undisturbed free-fall curtain shape are implicitly assumed to be smaller than the perturbations to the curtain shape. The disturbance function G' determines how the curtain thickness varies along the centerline described by F' (Fig. 2). Such variations are, by definition, varicose and are independent of any deflection of the curtain centerline. Then, the local total thickness of the curtain during a varicose response, h_t, is precisely given by the superposition of the free-fall thickness, $h(x)$ given by (14b), and G' as

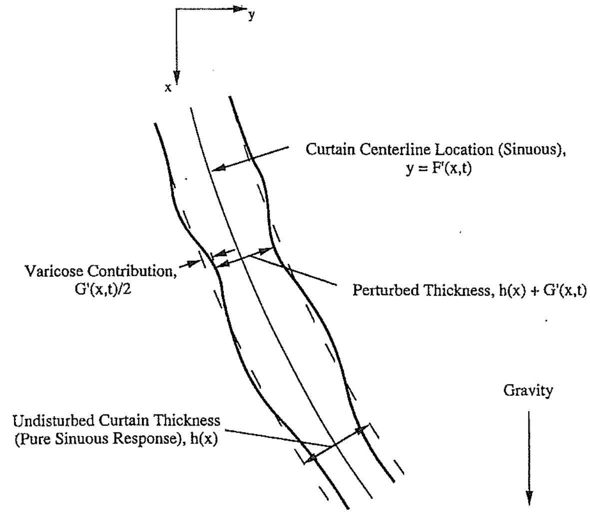

FIG. 2. Portion of the side view of the curtain shown in Fig. 1, indicating the superimposed sinuous and varicose responses of the curtain to an imposed disturbance. To the order of the approximations used in deriving the sinuous and varicose governing equations, the solution for the undisturbed curtain thickness, $h(x)$, is given by (14b).

$$h_t=h(x)+G'(x,t). \tag{26}$$

Based on the above discussion, Eq. (26) is valid, even when the curtain centerline deflects, i.e., $y=F'(x,t)\neq 0$ (Fig. 2). Inspection of (25a) and (25b) shows that pressure disturbances are translated into a deflection of the curtain centerline, since the applied pressures appear in the sinuous equation (25a); there are no pressure terms evident in the varicose equation (25b). In reality, pressure variations do interact with the varicose response, but they are higher-order effects [i.e., $O(\epsilon^2)$ or higher, according to our asymptotic expansion]. Consequently, we anticipate that it is difficult to excite a varicose curtain response due to pressure disturbances within the range of validity of our asymptotics. Flow disturbances in the curtain, which are related to the perturbed velocity potential, are also incurred at $O(\epsilon^2)$ according to the linearization (18a).

Before leaving this section, we find it useful to obtain alternative forms of the derived sinuous governing equation (25a). To this end, we utilize the expression (9a) as a variable transformation between $\bar{x}$ and $\bar{u}$. Substituting (9a) into (25a), we obtain the following alternative form of the dimensionless equation governing the curtain centerline:

$$\left(\frac{\partial}{\partial\bar{t}}+\frac{\partial}{\partial\bar{u}}\right)^2\bar{F}'-2W\frac{\partial}{\partial\bar{u}}\left(\frac{1}{\bar{u}}\frac{\partial\bar{F}'}{\partial\bar{u}}\right)$$
$$=\frac{\bar{u}(\bar{P}_2-\bar{P}_1)}{\epsilon^2},\quad \text{where } \bar{F}'\ll 1,\quad \bar{P}_2-\bar{P}_1\ll 1. \tag{27}$$

In obtaining (27), we have utilized the assumed forms (18d) to eliminate the occurrence of the imposed pressure perturbations $\overline{P}_1'$ and $\overline{P}_2'$; as indicated in (27), an equivalent statement to the elimination of these perturbed quantities is that the difference between the front and back face imposed pressures $\overline{P}_1(\overline{x},\overline{t})$ and $\overline{P}_2(\overline{x},\overline{t})$ is small. In fact, it is apparent by inspection of (27) that it is necessary for $\overline{P}_2-\overline{P}_1=O(\epsilon^2)$ or smaller to yield $\overline{F}'\ll 1$, which is entirely equivalent to the expansion (18d). The utility of the form (27) is clearly seen for cases where $W=0$ (i.e., surface tension effects are negligible in the curtain), for which the equation has constant coefficients and is straightforward to solve analytically. For convenience, the sinuous equation (27) is expressed in dimensional form as

$$\left(\frac{\partial}{\partial t}+g\,\frac{\partial}{\partial u}\right)^2 F'-\frac{2\sigma g^2}{\rho q}\,\frac{\partial}{\partial u}\left(\frac{1}{u}\,\frac{\partial F'}{\partial u}\right)=\frac{(P_2-P_1)u}{\rho q},\qquad(28)$$

where u is the free-fall speed given by (1a); the dimensional form of the varicose governing equation is given by (25b) without the overbars. General analytical solutions of the varicose equation (25b) are easily obtained and are provided in the Appendix.

IV. DISCUSSION OF VALIDITY OF EQUATIONS VIA COMPARISON WITH CITED LITERATURE RESULTS

We now demonstrate that the derived equations yield results that are consistent with those of the previously cited literature in Sec. I. We begin by noting that the sinuous equation (28) can be used to model the response of a curtain to a time-*independent* applied pressure. The steady case where a constant pressure drop is applied to a curtain of infinite width has been previously studied by Finnicum *et al.*,[8] as cited in Sec. I. In contrast to the asymptotic approach in Sec. III, Finnicum *et al.* utilizes a local macroscopic momentum balance to derive a nonlinear governing equation. The experiments of Finnicum *et al.* show excellent agreement between theoretical predictions of experimental curtain shapes. The time-independent form of the sinuous equation (28) is *identical* to that derived by Finnicum *et al.*, once their equation is linearized for small deflections in the curtain centerline.

We now consider the types of traveling wave solutions the sinuous and varicose equations admit to afford comparison with the results of Lin.[10] Recall that Lin determined wave speeds and growths for waves traveling in the curtain, utilizing Fourier mode analysis simplified for long wavelengths. Structurally, the derived sinuous equation (25a) is hyperbolic, provided $W>0$, based on standard classification methods of linear partial differential equations. In second-order hyperbolic systems, all disturbances propagate through the curtain along two characteristic lines that define trajectories in the $x-t$ domain. The two equations governing the characteristic lines are given by

$$\frac{d\overline{x}}{d\overline{t}}=\overline{u}\pm(2W\overline{u})^{1/2},\qquad(29)$$

which are interpreted as the speeds of propagation of information in the domain. These should correspond to the speed at which waves move through the curtain. Indeed, these speed expressions correspond precisely to those obtained for sinuous waves when linear stability theory and Fourier mode analyses (simplified for long wavelengths) are employed, as predicted by Lin;[10] this gives us confidence as to the validity of the sinuous equation (25a). Note, however, that the curtain stability information (wave growth or damping) contained in Lin's analysis, associated with the sinuous wave propagation, is not contained in the equation (25a), since these effects are related to the curtain viscosity; as shown by Lin, the damping or growth in the curtain is extremely small for sinuous waves at long wavelengths, which further justifies the neglect of viscosity in the current analysis. We note that the agreement between predictions of (25a) and Lin's traveling sinuous waves indicates that simplifications to the potential flow equations in our analysis afforded by the small parameter ϵ in (3b) (i.e., the assumption of a "long and thin" curtain) is identical in nature to utilizing a long-wavelength analysis in the Fourier domain.

Continuing our comparison with Lin's traveling wave results, we now focus on the waves admitted by the varicose equation (25b). This equation is first order, and information travels along a single characteristic line at the free-fall speed u (see the Appendix); that is, the speed of waves is equal to the free-fall speed. Inspection of Lin's varicose solution shows that when viscosity is neglected (as in the current case), the dimensionless wave speed, c/V, can be written as

$$c/V\sim u/V\pm\alpha\left(\frac{\sigma}{2\rho q u}\right)^{1/2},\qquad(30)$$

where $\alpha=2\pi d/\lambda$ is the dimensionless wave number arising through his Fourier mode analysis, V is the velocity at the top of the curtain, and u is the free-fall speed given by (1a). The parameter α in Lin's formulation can be viewed as a small aspect ratio, identical in nature to the small aspect ratio ϵ given in (3b) that is used to simplify the current analysis. In deriving the varicose equation (25b), we performed an expansion, assuming that the quantity in the brackets in (30) was order 1 in the limit of small aspect ratio; inspection of the wave speed relation c given in (30) indicates that the bracketed term is thus a higher-order effect since $\alpha\ll 1$. The equation (30) thus yields the fact that $c\sim u$, in agreement with wave speeds predicted by the varicose equation (25b).

Further evidence for the validity of the varicose equation (25b) comes from direct comparison with the equation of Ramos,[7] which is valid for viscous flows. In the limit of large Reynolds number and under steady conditions, his equation (56) leads directly to a free-fall velocity in the curtain, and his equation (57) yields the lowest-order curtain thickness given in our paper. By further linearizing his time-dependent curtain thickness and velocity fields about this inviscid base flow, and noting that, according to the asymptotic ordering in our analysis, perturbations to the velocity field are of a higher order than those to the film thickness, his unsteady equation (57) is identical to our varicose equation (25b).

Note that the above arguments regarding the propagation of sinuous waves can also be used to formally justify the placement and number of boundary conditions in the steady-state problem of Finnicum *et al.*[8] This is because any steady problem can be viewed as the long-time behavior of a transient problem (in which the transient effects have died out). Thus, the location and specification of the boundary conditions in Finnicum *et al.* must be consistent with the ultimate long-time behavior of a transient problem, or else the steady problem will not be obtained in the limit of large time. In hyperbolic equations, the number of boundary conditions must be consistent with the number of characteristics, and the direction of the flow of information must be consistent with the direction of motion of the characteristics. As discussed in Sec. I, the placement of boundary conditions in the steady problem of Finnicum *et al.* depends upon the value of the Weber number, W. According to Finnicum *et al.*, for $W<\frac{1}{2}$, two boundary conditions can be placed at the top of the curtain; this is self-consistent with the fact that according to (29), both characteristics always propagate down the curtain, since $\bar{u}\geqslant 1$ in the curtain and $\bar{u}$ increases with increasing $\bar{x}$. On the other hand, if $W>\frac{1}{2}$, and $\bar{u}=2W$ in the curtain, Finnicum *et al.* show that one constraint can be applied at the top of the curtain, and a second must be applied at the location where $\bar{u}=2W$. According to (29), the characteristic corresponding to the plus sign always propagates downward in the curtain, which is consistent with a constraint being applied at the top of the curtain. However, for the domain $1\leqslant\bar{u}<2W$, the characteristic corresponding to the minus sign in (29) propagates upward toward the top of the curtain. Beyond this point, where $\bar{u}>2W$, the characteristic propagates downward. This indicates the necessity of applying a boundary condition at the location where $\bar{u}=2W$ in the curtain, again consistent with the boundary condition placement of Finnicum *et al.*; it turns out that a weak constraint (boundedness of the interface solution) is specified by Finnicum *et al.* at this location.

The comparisons described above between the derived equations (25) and the known results in the literature cited in Sec. I provide evidence of their validity. In part 2 of this paper, we provide experimental verification of the sinuous equation predictions for time-dependent disturbances. In the experiments, periodic disturbances are generated via electrocapillarity and are detected via optical reflection. Both these techniques are noncontact and allow for precise measurement of amplitude and phase to be made for linear disturbances. Measurements are presented as a function of location x in the curtain for a range of Weber numbers and frequencies, and show that the sinuous equation accurately predicts the liquid interface shape.

V. SUMMARY

Approximate equations have been derived that govern the time-dependent response of a liquid curtain to ambient pressure disturbances. Examination of the previous literature indicates that the assumption of potential flow is valid downstream of the slot entrance region, provided that (2c) is satisfied (i.e., viscous effects are small). Assuming that the curtain is long and thin, as characterized by the small values of the aspect ratio ϵ in (3b), a steady-state base flow has been determined. The steady analysis reveals that the lowest-order approximation to the curtain flow is the free-fall velocity u as given by (1a); the corresponding lowest-order curtain thickness, h, is as expected from conservation of the volumetric flow rate per width, q, where $h=q/u$. The $O(\epsilon^2)$ corrections indicate that the velocity and internal pressure profiles are only slightly curved and also provide an associated adjustment to the local curtain thicknesses. The magnitude of the corrections decreases with distance down the curtain.

By assuming that the disturbances to the curtain are small, the time-dependent equations have been linearized about the approximated base flow. The approximate nature of the base flow necessitates a particularly careful ordering of terms to assure the validity of the linearization process. The derived equations separately govern the location of the curtain centerline and thickness variations in the curtain; these are given, respectively, by (25a) and (25b). Previous literature results regarding wave propagation in the curtain, as well as its steady-state deflection due to an applied pressure drop, are predicted via the derived equations. Finally, it has been shown that to the order of the approximations considered in this paper, pressure disturbances induce a deflection of the curtain without any associated thickness variations; that is, the front and back interfaces are perturbed in precisely the same way to preserve the local thickness of the undisturbed curtain.

ACKNOWLEDGMENT

We would like to thank Dr. Kenneth J. Ruschak for providing us with a preliminary mathematical analysis of time-dependent curtain motion, as well as many important discussions throughout the course of this work.

APPENDIX: ANALYTICAL SOLUTIONS TO THE VARICOSE EQUATION

We now consider the general form of solutions to the varicose equation (25b). Inspection of (25b) shows that it is a first partial differential equation, which can be solved via the standard method of characteristics. We consider the general solution for two cases in what follows; the analysis for these cases will be cast in dimensional form. Note that the dimensional form of the varicose equation is given by (25b), but without the overbars.

1. Case 1: Thickness variation is prescribed at a given location

The curtain thickness is disturbed with a prescribed variation $R(t)$ at a given location x_0 in the curtain as

$$G'=R(t), \quad \text{at } x=x_0.$$

The dimensional solution of (25b) is given by

$$G'=\frac{u_0}{u}R(\eta), \tag{A1a}$$

where η is a characteristic defined as

$$\eta=t+\frac{(u_0-u)}{g}. \tag{A1b}$$

In (A1), u is the free-fall velocity given by (1a), and u_0 is the curtain speed at the location x_0. Disturbance information carried along a given characteristic, $\eta=\text{const}$, propagates down the curtain at the local free-fall speed u, which can be seen by direct differentiation of (A1b). For a given value of η, the function $R(\eta)$ in (A1a) is constant; the solution (A1) thus describes a translating thickness variation whose amplitude decreases with increasing distance down the curtain as $1/u$.

2. Case 2: Thickness variation is prescribed at a given time

The curtain thickness is disturbed with a prescribed variation $S(x)$ at a given time t_0 in the curtain as

$$G'=S(x), \quad \text{at } t=t_0.$$

The dimensional solution of (25b) is given by

$$G'=\frac{u(\zeta)}{u}S(\zeta), \tag{A2a}$$

where ζ is a characteristic defined as

$$\zeta=t+\frac{[u+g(t-t_0)]^2-V^2}{2g}. \tag{A2b}$$

In (A2), u is the free-fall velocity given by (1a); $u(\zeta)$ is (1a) evaluated at $x=\zeta$. Although the solution form (A2) is different from (A1) in the previous case, the physical interpretation of (A2) is essentially the same.

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