Gravity-driven instability of a thin liquid film underneath a soft solid

S. H. Lee,1 K. L. Maki,2 D. Flath,3 S. J. Weinstein,4 C. Kealey,5 W. Li,3 C. Talbot,6 and S. Kumar1
1Department of Chemical Engineering and Materials Science, University of Minnesota, Minneapolis, Minnesota 55455, USA
2School of Mathematical Sciences, Rochester Institute of Technology, Rochester, New York 14623, USA
3Department of Mathematics, Statistics, and Computer Science, Macalester College, St. Paul, Minnesota 55105, USA
4Department of Chemical Engineering, Rochester Institute of Technology, Rochester, New York 14623, USA
5Department of Mathematics, Beloit College, Beloit, Wisconsin 53511, USA
6Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269, USA

(Received 13 August 2014; published 17 November 2014)

The gravity-driven instability of a thin liquid film located underneath a soft solid material is considered.

The present paper has two objectives. The first is to present a systematic derivation of the governing equations and boundary conditions that describe the liquid-solid interface. In doing so, we will begin with a Lagrangian description of the solid motion and carefully convert it to an Eulerian description. We will show that even when the displacement gradients are small (the small-strain limit), the continuity-of-velocity boundary condition at the liquid-solid interface is more complicated than has previously been assumed. Our systematic conversion allows us to make clear the conditions under which the commonly used simplified version of this boundary condition is valid.

$\rho \left( \frac{\partial \mathbf{v}^\ell}{\partial t} + \mathbf{v}^\ell \cdot \nabla \mathbf{v}^\ell \right) = \nabla \cdot \mathbf{T}^\ell + \rho g \mathbf{k}$,

$\nabla \cdot \mathbf{v}^\ell = 0$,

where $\rho$ is the density of the liquid, $\mathbf{v}^\ell$ is the velocity vector of the liquid, $\mathbf{T}^\ell$ is the stress tensor of the liquid, and $g$ is the gravitational acceleration. The operator $\mathbf{V} = i \partial / \partial x + k \partial / \partial z$, where $i$ and $k$ denote the unit vectors in the $x$ and $z$ directions, respectively. The liquid is modeled as Newtonian so the total stress tensor is given by

$\mathbf{T}^\ell = -p^\ell I + \eta^\ell [\nabla \mathbf{v}^\ell + (\nabla \mathbf{v}^\ell)^T]$,

where the pressure is denoted by $p^\ell$, $I$ is the identity tensor, $\eta^\ell$ is the liquid viscosity, and the superscript $T$ denotes transpose. At the liquid-air interface $z = f(x, r)$, we impose a
B. Gel motion

The stress and strain in a solid are most naturally described from a Lagrangian perspective, as the strain invoked is determined via reference to the undeformed locations in the body. However, the motion of the liquid is more naturally described in Eulerian form, where the motion of arbitrary material bodies of finite volume can be tracked in time, and the formalism of continuum mechanics can be applied to generate differential field equations. In addition, the gel and liquid motions are intimately coupled. In what follows, we present general governing equations for the gel from an Eulerian perspective. The stress in the gel is indeed described in Lagrangian form but then is carefully mapped back to the Eulerian perspective so it can be inserted into the Eulerian equations of motion.

From an Eulerian perspective, the motion of the gel is governed by statements of momentum and mass conservation:

$$\rho' \left( \frac{\partial \mathbf{v}^s}{\partial t} + \mathbf{v}^s \cdot \nabla \mathbf{v}^s \right) = \nabla \cdot \mathbf{T}^s + \rho' g \mathbf{k}$$

$$\nabla \cdot \mathbf{v}^s = 0,$$

where $\rho'$, $\mathbf{v}^s$, and $\mathbf{T}^s$ are, respectively, the density, velocity vector, and stress tensor of the gel. For simplicity, we adopt the approach of much prior literature and model the gel as a Kelvin-Voigt material. In the Kelvin-Voigt model, the motion of the gel is represented by a viscous damper and purely elastic spring connected in parallel. Although here we focus on the small-strain limit, we note that accounting for the effect of finite strains in the gel requires the use of a nonlinear constitutive model \cite{2,3}. Before considering the expression for $\mathbf{T}^s$, we review mass conservation from a Lagrangian perspective; this is necessary to carefully establish conditions under which the approximations used in the analysis to follow are valid.

The dynamics of the gel are characterized by the Lagrangian displacement field, the measurement of the deviation of the gel from its unstressed state. The reference configuration has independent spatial variables $(X, Z)$ to characterize the material particles in the reference (i.e., unstressed) frame. In the deformed state, the $(x, z)$ spatial location of the material particle $(X, Z)$ in the reference configuration is given by:

$$x = X + U_x(X, Z, t) \quad \text{and} \quad z = Z + U_z(X, Z, t),$$

where $U_x$ and $U_z$ are the Lagrangian displacements in the $x$ and $z$ directions, respectively.

In the limit of small strains, the incompressibility of the gel is expressed as

$$\frac{\partial U_x}{\partial X} + \frac{\partial U_z}{\partial Z} = 0.$$  

To see why, we consider an initially rectangular gel element in the $XZ$ plane with sides $\Delta X$ and $\Delta Z$ aligned with and parallel to the $X$ and $Z$ axes, respectively. The area of this element is thus $\Delta X \Delta Z$. This rectangle is deformed in the $xz$ plane as shown in Fig. 2 and has vertices given by

$$P_1 = [X + U_x(X, Z, t), Z + U_z(X, Z, t)],$$

$$P_2 = [X + U_x(X, Z + \Delta Z, t), Z + \Delta Z + U_z(X, Z + \Delta Z, t)],$$

$$P_3 = [X + \Delta X + U_x(X + \Delta X, Z, t), Z + \Delta Z + U_z(X + \Delta X, Z + \Delta Z, t)],$$

$$P_4 = [X + \Delta X + U_x(X + \Delta X, Z + \Delta Z, t), Z + U_z(X + \Delta X, Z, t)].$$

The area of the stretched gel element is given by the cross product $\mathbf{P}_1 \mathbf{P}_2 \times \mathbf{P}_3 \mathbf{P}_4$. If we ignore quadratic terms in the Taylor expansions as $\Delta X$ and $\Delta Z$ approach zero, then the vectors $\mathbf{P}_1 \mathbf{P}_2$ and $\mathbf{P}_3 \mathbf{P}_4$ are given by

$$\mathbf{P}_1 \mathbf{P}_2 = \left[ \Delta Z \frac{\partial U_x}{\partial Z}(X, Z, t), \Delta Z + \frac{\partial U_z}{\partial Z}(X, Z, t) \right],$$

$$\mathbf{P}_3 \mathbf{P}_4 = \left[ \Delta X + \Delta X \frac{\partial U_x}{\partial X}(X, Z, t), \Delta X \frac{\partial U_z}{\partial X}(X, Z, t) \right].$$

Thus, the area of the stretched gel element is

$$\begin{vmatrix} \frac{\partial U_x}{\partial X} & \frac{\partial U_z}{\partial Z} \\ \frac{\partial U_z}{\partial X} & \frac{\partial U_x}{\partial Z} \end{vmatrix} \Delta X \Delta Z.$$  

For the volume to be conserved, i.e., the area remains as $\Delta X \Delta Z$, we must have

$$\frac{\partial U_x}{\partial Z} + \frac{\partial U_z}{\partial X} + \frac{\partial U_z}{\partial Z} \frac{\partial U_x}{\partial Z} - \frac{\partial U_z}{\partial Z} \frac{\partial U_x}{\partial Z} = 0.$$  

In this work, we will apply the small-strain approximation, which requires that the displacement gradients be small.
compared to unity, i.e.,
\[
\frac{\partial U_x}{\partial X} \ll 1, \quad \frac{\partial U_x}{\partial Z} \ll 1, \quad \frac{\partial U_z}{\partial X} \ll 1, \quad \text{and} \quad \frac{\partial U_z}{\partial Z} \ll 1.
\] (10)
This approximation allows products of displacement gradients to be dropped from Eq. (9) and leads to the incompressibility condition stated in Eq. (7).

The strain, or measure of deformation in the gel, is given by spatial derivatives of the displacement fields. The components of the resulting Lagrangian infinitesimal-strain tensor \( L \) are given by
\[
L_{xx} = \frac{\partial U_x}{\partial X}, \quad L_{xz} = L_{zx} = \frac{1}{2} \left( \frac{\partial U_x}{\partial Z} + \frac{\partial U_z}{\partial X} \right), \quad \text{and} \quad L_{zz} = \frac{\partial U_z}{\partial Z}. \]
(11)
To characterize the purely elastic behavior, we use Hooke’s law where the stress tensor to be used in conjunction with Eq. (4). The constitutive law is linear from the Lagrangian perspective.

Physically, the Lagrangian description fixes attention on specific particles of the gel, whereas the Eulerian description concerns itself with a particular region of the space occupied by the gel.

To write down general formulas mapping the Lagrangian derivatives into Eulerian derivatives, we assume we have a generic function with \( H(X,Z,t) = h(x(X,Z,t), Z(X,Z,t), t) \). Using the relationships \( x = X + U_x(X,Z,t) \) and \( z = Z + U_z(X,Z,t) \), we find
\[
\frac{\partial h}{\partial X} = \frac{\partial h}{\partial x} \left( 1 + \frac{\partial U_x}{\partial X} \right) + \frac{\partial h}{\partial z} \left( \frac{\partial U_z}{\partial X} \right),
\]
(16)
where the subscript indicates what independent variables are being held fixed when derivatives are taken. To generate equations for the Lagrangian partial derivatives of \( U_x \) and \( U_z \) in terms of Eulerian partial derivatives of \( u_x \) and \( u_z \), we use the generic formula Eq. (16) with \( H = U_x \) and \( h = u_x \) as well as \( H = U_z \) and \( h = u_z \). We find
\[
\left. \frac{\partial U_x}{\partial X} \right|_{Z,t} = \frac{A}{J}, \quad \left. \frac{\partial U_z}{\partial X} \right|_{Z,t} = \frac{B}{J}, \quad \left. \frac{\partial U_x}{\partial Z} \right|_{X,t} = \frac{C}{J}, \quad \left. \frac{\partial U_z}{\partial Z} \right|_{X,t} = \frac{D}{J},
\]
where
\[
A = \frac{\partial u_x}{\partial x}_{x,t} - \frac{\partial u_x}{\partial z}_{x,t} \frac{\partial u_z}{\partial x}_{x,t} + \frac{\partial u_z}{\partial x}_{x,t},
B = \frac{\partial u_x}{\partial x}_{x,t},
J = 1 - \frac{\partial u_x}{\partial z}_{x,t} - \frac{\partial u_z}{\partial x}_{x,t} \frac{\partial u_z}{\partial z}_{x,t} + \frac{\partial u_z}{\partial z}_{x,t},
\]
(17)
An identical process is used to map the partial derivatives with respect to \( Z \):
\[
\left. \frac{\partial U_x}{\partial Z} \right|_{X,t} = \frac{C}{J}, \quad \left. \frac{\partial U_z}{\partial Z} \right|_{X,t} = \frac{D}{J}, \quad \left. \frac{\partial U_x}{\partial x} \right|_{Z,t} = \frac{C}{J}, \quad \left. \frac{\partial U_z}{\partial x} \right|_{Z,t} = \frac{D}{J}.
\]
(18)
We will now formally argue that the small-strain approximation in the Lagrangian coordinates implies the small-strain approximation in the Eulerian coordinates. To do so, we assume that terms quadratic in the Eulerian derivatives are small and utilize the method of dominant balance. With this assumption, we obtain
\[
\frac{\partial u_x}{\partial x}_{x,t} \sim \frac{\partial U_x}{\partial X}_{Z,t}, \quad \frac{\partial u_z}{\partial x}_{x,t} \sim \frac{\partial U_z}{\partial X}_{Z,t}, \quad \frac{\partial u_z}{\partial z}_{x,t} \sim \frac{\partial U_z}{\partial Z}_{X,t}, \quad \frac{\partial u_z}{\partial z}_{x,t} \sim \frac{\partial U_z}{\partial Z}_{X,t}.
\]
(19)
Substituting these results back into Eqs. (17) and (18), and invoking incompressibility of the gel Eq. (7) in \( J \) [defined in
Eq. (17)] shows that indeed quadratic terms in the Eulerian derivatives are equivalent in order to quadratic terms in the Lagrangian derivatives. Thus, Eqs. (19) and (20) are asymptotically consistent with the small-strain approximation used in the Lagrangian framework.

Next, we determine the consequence of the infinitesimal-strain approximation on the time derivatives. Specifically, we again assume we have a generic function with strain approximation on the time derivatives. Thus, Eqs. (19) and (20), and the incompressibility assumption, Eqs. (23) and (24), and the incompressibility framework to an Eulerian one.

With the above results, the components of the Eulerian total stress tensor are given by

\[
T^n_{xx} = -p^n + 2 \eta^n \frac{\partial v^n}{\partial x} + 2 E \frac{\partial u^n}{\partial x},
\]

\[
T^n_{xz} = T^n_{zx} = \eta^n \left( \frac{\partial v^n}{\partial x} + \frac{\partial v^t}{\partial z} \right) + E \left( \frac{\partial u^n}{\partial x} + \frac{\partial u^t}{\partial z} \right),
\]

\[
T^n_{zz} = -p^n + 2 \eta^n \frac{\partial v^n}{\partial z} + 2 E \frac{\partial u^n}{\partial z},
\]

where

\[
v^n = \frac{\partial u^n}{\partial t},
\]

\[
v^t = \frac{\partial u^t}{\partial t}.
\]

The expressions for \(v^n\) and \(v^t\) from a Lagrangian perspective are given by Eq. (13). Here, the constitutive law is linear from an Eulerian perspective, but it would have been nonlinear had the terms in Eqs. (23) and (24) involving products of a displacement gradient and a time derivative been retained. In addition, we note that Eqs. (30) and (31) are the expressions used in prior work, where they are simply written down (e.g., Refs. [4–8]). However, the order-of-magnitude arguments given above make clear the assumptions under which these expressions are valid.

We now state the remaining boundary conditions. At the rigid plane \(z = -HR\), the displacements are zero; that is, the deformable gel is perfectly attached to the horizontal plane. At the gel-liquid interface, \(z = f^t(x, t)\), the normal and tangential stresses in the gel and liquid are balanced:

\[
n^n \cdot T^n - n^n \cdot T^n + 2 \sigma^t \mathcal{H} n^n = 0,
\]

where \(n^n\) is the normal vector to the gel-liquid interface that points into the liquid, \(\sigma^t\) is the surface tension of the gel-liquid interface, and \(\mathcal{H}\) is the mean curvature of the gel-liquid interface. The velocities in the gel and the liquid are equal, which enforces both the no-slip and no-penetration conditions:

\[
u^n_x = u^t_x \quad \text{and} \quad v^n_z = v^t_z.
\]
In addition, there is a kinematic condition describing the location of the gel-liquid interface. In what follows, we simplify the calculations by studying the limit in which the gel has no viscosity. Thus, \( \eta^g = 0 \) and we refer to the gel as a solid.

### C. Leading-order equations

The liquid thickness \( R \) is taken as a characteristic length scale in the vertical direction. We denote the characteristic length scale in the horizontal direction as \( \lambda \) and assume that \( \epsilon = R/\lambda \ll 1 \), allowing us to focus on long-wavelength perturbations to the system \([1,9]\). Specific choices could be made for \( \lambda \) (e.g., the instability wavelength), but here we leave \( \lambda \) arbitrary for generality. The governing equations are made dimensionless with the following scalings:

\[
x' = \frac{x}{\lambda}, \quad z' = \frac{z}{\epsilon \lambda}, \quad t' = \frac{t}{\lambda/\sqrt{\nu}}, \tag{34}
\]

\[
(v^l)' = \frac{v^l}{V}, \quad (v^s)' = \frac{v^l}{\epsilon \lambda}, \quad (p^l)' = \frac{p^l}{\eta^l V/(\epsilon^2 \lambda)}, \tag{35}
\]

\[
u^l = \frac{\nu^l}{\lambda}, \quad u^l = \frac{u^l}{\epsilon \lambda}, \quad (p^l)' = \frac{p^l}{\eta^l V/(\epsilon^2 \lambda)}, \tag{36}
\]

where \( V \) is a characteristic velocity scale in the \( x \) direction. Note that the scaling of time is consistent with the order-of-magnitude argument given in the previous section. It should also be recognized that in order to be consistent with both the small-strain limit and lubrication theory, we require \( \delta/R \ll 1 \) and \( R/\lambda \ll 1 \), where \( \delta \) is a characteristic displacement. In what follows, all variables are dimensionless and the prime superscript is omitted.

In the limit \( \epsilon \to 0 \), the following leading-order dimensionless equations are obtained in the liquid:

\[
0 = -\frac{\partial p^l}{\partial x} + \frac{\partial^2 v^l}{\partial z^2}, \tag{37}
\]

\[
0 = -\frac{\partial p^l}{\partial z} + \epsilon \rho G, \tag{38}
\]

\[
0 = \frac{\partial v^l}{\partial t} + \frac{\partial v^l}{\partial x}, \tag{39}
\]

where \( G = \rho^l g R^2/\eta^l V \) reflects the physical balance between the gravitational and viscous forces (from the liquid). We note that one could define a rescaled version of \( G \) equal to \( \epsilon G \) if desired. At the liquid-air interface \( z = f^l(x,t) \), we have

\[
\frac{\partial f^l}{\partial t} + v^l \frac{\partial f^l}{\partial x} = v^l, \tag{40}
\]

\[
p^a - p^l = S^l \frac{\partial^2 f^l}{\partial x^2}, \tag{41}
\]

\[
\frac{\partial v^l}{\partial z} = 0, \tag{42}
\]

where \( S^l = \epsilon^3 \sigma^a/\eta^l V \) is the ratio of the forces due to the liquid-air interfacial tension relative to those due to the liquid viscosity, and \( \rho^l \) is the pressure of the air, taken to be zero.

In the solid, we have

\[
-\frac{\partial p^s}{\partial x} + \bar{E} \frac{\partial^2 u^s}{\partial x^2} = 0, \tag{43}
\]

\[
-\frac{\partial p^s}{\partial z} + \epsilon \rho G = 0, \tag{44}
\]

\[
\frac{\partial u^s}{\partial x} + \frac{\partial u^s}{\partial z} = 0, \tag{45}
\]

where \( \bar{E} = E\lambda/\eta^l V \) measures the relative significance of elastic to viscous forces (from the liquid), and \( \rho = \rho^l/\rho^g \) is the solid-liquid density ratio. At the surface of the horizontal plane \( z = -H \),

\[
u^s(x, -H, t) = 0 \quad \text{and} \quad u^s(x, -H, t) = 0. \tag{46}
\]

Finally, at the liquid-solid interface \( z = f^l(x,t) \), the kinematic condition, continuity-of-velocity boundary conditions, and continuity-of-stress boundary conditions become

\[
\frac{\partial f^s}{\partial t} + v^l \frac{\partial f^l}{\partial x} = v^l, \tag{47}
\]

\[
\frac{\partial u^s}{\partial t} = v^s, \tag{48}
\]

\[
\frac{\partial u^s}{\partial t} = v^l, \tag{49}
\]

\[
p^l - p^s = S^l \frac{\partial^2 f^l}{\partial x^2}, \tag{50}
\]

\[
\frac{\partial v^l}{\partial z} = \bar{E} \frac{\partial u^s}{\partial z}, \tag{51}
\]

where \( S^l = \epsilon^3 \sigma^a/\eta^l V \) is the ratio of the forces due to the gel-liquid interfacial tension relative to those due to the liquid viscosity.

### III. LINEAR STABILITY ANALYSIS

The base state of the present system corresponds to flat liquid-solid and liquid-air interfaces that are located at \( z = 0 \) and \( z = 1 \), respectively. The liquid is at rest and the solid is undeformed in the base state. The velocity and pressure distributions in the base state, denoted by an overbar, are solutions to the leading-order system Eqs. (37)–(51) with no free-surface or interfacial deformations \((f^l = 0 \text{ and } f^s = 1)\):

\[
\bar{v}^l = 0, \quad \bar{v}^s = 0, \quad \text{and} \quad \bar{p}^l = \epsilon G(z - 1). \tag{52}
\]

The base-state displacements in the solid are

\[
\bar{u}^s = 0, \quad \bar{u}^l = 0, \quad \text{and} \quad \bar{p}^s = \epsilon G(\rho z - 1). \tag{53}
\]

We study the stability of the base state to small-amplitude perturbations. To each variable, a perturbation of the form

\[
F^l(x, z, t) = \tilde{F}(z)e^{i(kx - \omega t)}, \tag{54}
\]

where \( \tilde{F}(z) \) is a complex-valued eigenfunction, \( k \) is a wavenumber, and \( \omega \) is a complex-valued growth rate, is added to the base state and substituted into the leading-order system. With this choice for the normal mode, the perturbation
quantities grow and instability occurs when the imaginary part of \( \omega \), denoted by \( \text{Im}[\omega] \), is positive. Note that the quantities and \( f^\ell(x,t) \) and \( f^s(x,t) \) do not depend on \( z \), so the quantities \( f^\ell \) and \( f^s \) are constants.

The governing equations for the perturbation quantities in the liquid and the solid, respectively, are given by

\[
0 = -ik \tilde{p}^\ell + \frac{d^2 \tilde{v}^\ell}{dz^2},
\]

(55)

\[
0 = -\frac{d \tilde{p}^s}{dz},
\]

(56)

\[
0 = ik \tilde{v}^\ell + \frac{d \tilde{v}^\ell}{dz},
\]

(57)

\[
0 = -ik \tilde{p}^s + \bar{E} \frac{d^2 \tilde{u}_x}{dz^2},
\]

(58)

\[
0 = -\frac{d \tilde{p}^s}{dz},
\]

(59)

\[
0 = ik \tilde{u}_x + \frac{d \tilde{u}_x}{dz}.
\]

(60)

At the liquid-air interface, \( z = 1 \), we apply a domain perturbation method to find

\[
-i\omega \tilde{f}^\ell = \tilde{v}^\ell,
\]

(61)

\[
-\epsilon G \tilde{f}^s - \tilde{p}^s = -k^2 S^s \tilde{f}^s,
\]

(62)

\[
\frac{d \tilde{v}^\ell}{dz} = 0,
\]

(63)

where the first equation is the kinematic condition, the second is the normal force balance, and the third is the tangential force balance. The domain perturbation method involves replacing each variable as the sum of its base state value and a perturbation of the form of Eq. (54). The boundary conditions are then expanded in a Taylor series around the location of the unperturbed interface and only terms linear in the perturbation quantities are retained [10]. Note that the perturbation parameter associated with the liquid-air interface, \( \tilde{f}^\ell \), is a constant.

Similarly, at the liquid-solid interface, \( z = 0 \), the domain perturbation method gives the following five boundary conditions:

\[
-i\omega \tilde{f}^s = \tilde{v}^s,
\]

(64)

\[
(1 - \rho)\epsilon G \tilde{f}^s + \tilde{p}^s = -k^2 S^s \tilde{f}^s,
\]

(65)

\[
\frac{d \tilde{v}^s}{dz} = \bar{E} \frac{d \tilde{u}_x}{dz},
\]

(66)

\[
-i\omega \tilde{u}_x = \tilde{v}^s,
\]

(67)

\[
-i\omega \tilde{u}_z = \tilde{v}^s,
\]

(68)

where the first equation is the kinematic condition, the second two equations are the force balances, and the final two equations are the continuity-of-velocity boundary conditions in the \( x \) and \( z \) directions, respectively. Again, the perturbation parameter associated with the liquid-air interface, \( \tilde{f}^\ell \), is a constant in the system above. Finally, to close the system,

\[
\tilde{u}_x(-H) = \tilde{u}_z(-H) = 0,
\]

(69)

at the rigid substrate. The system Eqs. (55)–(69) constitute the generalized eigenvalue problem that needs to be solved to complete the linear stability analysis.

The stability problem is solved analytically. From the governing equations for the liquid perturbation quantities, we have

\[
\tilde{v}^\ell(z) = \frac{ik \tilde{p}^\ell}{2} z^2 + c_1 z + c_2,
\]

(70)

\[
\tilde{v}^s(z) = \frac{k^2 \tilde{p}^s}{6} z^3 - \frac{ikc_1}{2} z^2 - ikc_2 z + c_3,
\]

(71)

FIG. 3. (Color online) (a) Growth rate versus wavenumber for \( H = 0 \) (dashed line) and \( H = 0.1 \) (solid line) with \( \bar{E} = 1 \), \( S^s = 1 \), \( S^\ell = 0 \), \( \epsilon = 0.1 \), and \( \rho = 1 \). The case \( H = 0 \) corresponds to a rigid substrate. (b) Growth rate versus wavenumber for \( \bar{E} = 1 \), \( G = 10 \), \( S^s = 1 \), \( S^\ell = 0 \), \( \epsilon = 0.1 \), \( \rho = 1 \), and different values of \( H: H = 0 \) (solid line), \( H = 0.1 \) (dashed line), \( H = 0.5 \) (dashed-dot line), \( H = 1 \) (circles), and \( H = 2 \) (triangles).
where \( \bar{p}^t \), \( c_1 \), \( c_2 \), and \( c_3 \) are constants. Applying the kinematic condition Eq. (61), we find

\[
c_3 = -i\omega f^t - \frac{k^2 \bar{p}^t}{6} + ikc_1 + ikc_2. \tag{72}
\]

The tangential stress condition Eq. (63) gives

\[
c_1 = -ik \bar{p}^t, \tag{73}
\]

and the normal stress condition Eq. (62) yields

\[
\bar{p}^t = \bar{f}^t (k^2 S^a - \epsilon G). \tag{74}
\]

Therefore, the perturbation liquid velocities are given by

\[
\bar{v}^t_s(z) = \frac{ik \bar{p}^t}{2} z^2 - ik \bar{p}^t z + c_2, \tag{75}
\]

\[
\bar{v}^t_z(z) = \frac{k^2 \bar{p}^t}{6} (z^3 - 1) + \frac{k^2 \bar{p}^t}{2} (1 - z^2) + ikc_2(1 - z) - i\omega \bar{f}^t, \tag{76}
\]

where \( \bar{p}^t \) is given by Eq. (74); these velocities are expressed in terms of unknown constants \( c_3 \) and \( \bar{f}^t \). Similarly, the perturbation quantities for the displacements in the solid are given by

\[
\bar{u}_s(z) = i k \bar{p}^t \left( \frac{z^2}{2} - \frac{H^2}{2} \right) + c_4 (z + H), \tag{77}
\]

\[
\bar{u}_z(z) = \frac{k^2 \bar{p}^t}{E} \left( \frac{z^3}{6} - \frac{H^2 z}{2} - \frac{H^3}{3} \right) - i k c_4 \left( \frac{z^2}{2} + Hz + \frac{H^2}{2} \right). \tag{78}
\]

where \( \bar{p}^t \) and \( c_4 \) are unknown constants.

We substitute the calculated perturbation quantities into the five liquid-solid boundary condition Eqs. (64)–(68) to obtain a linear system of equations for the unknown constants \( c_2, \bar{f}^t, \bar{f}^s, \bar{p}^t, \bar{u}_{s, \text{full}}, \) and \( c_4 \). The determinant of the linear system must be zero because we seek nontrivial solutions. The characteristic equation is a quadratic in the complex growth rate, where one root is found to always be zero. The imaginary part of the second root is examined to determine stability.

\[\text{IV. RESULTS}\]

The characteristic equation governing the growth rate is

\[
\left[ -k^2 \frac{H^3}{3} (D1 + D2) - k^2 H^2 (D1) - k^4 \frac{H^4}{12E} (D1)(D2) - \bar{E} \right] \omega^2 + \left[ -k^4 \frac{H^3}{9} (D1)(D2) - k^2 \frac{1}{3} \bar{E}(D1) \right] i \omega = 0, \tag{79}
\]

where

\[
D1 = k^2 S^a - \epsilon G, \quad D2 = k^2 S^a + (1 - \rho)\epsilon G.
\]

To further simplify matters, we set \( \rho = 1 \) (the density of the liquid equals the density of the solid) and \( S^a = 0 \) (there is no surface tension at the liquid-solid interface). Then, the nonzero root of the characteristic equation is

\[
\omega = -i \frac{-k^2 \frac{1}{3} \bar{E}(k^2 S^a - \epsilon G)}{-k^2 \frac{1}{3} \bar{E}(k^2 S^a - \epsilon G) - k^2 H^2 (k^2 S^a - \epsilon G) - k^2 H (k^2 S^a - \epsilon G) - \bar{E}}. \tag{80}
\]

Note that in the case where \( H = 0 \) (solid thickness is zero),

\[
\text{Im}[\omega] = -k^2 \frac{1}{3} (k^2 S^a - \epsilon G), \tag{81}
\]

which is the well-known result for the case where the solid is rigid [1].

Figure 3 shows how \( \text{Im}[\omega] \) varies as function of \( k, G, \) and \( H \). Figure 3(a) shows that the maximum growth rate and the range of unstable wavenumbers increase as \( G \) increases. Explicit expressions for the cutoff and most-dangerous wavenumbers are readily obtained,

\[
k_{\text{cut}} = \frac{\sqrt{\epsilon G}}{S^a}, \tag{82}
\]

\[
k_{\text{mcut}} = \frac{\sqrt{\epsilon G}}{2S^a}. \tag{83}
\]

These expressions do not depend on the thickness of the deformable layer. We note that setting the value of \( S^a \) determines the velocity scale \( V \), and our choice of \( S^a = 1 \) is consistent with lubrication theory [1,9].

The maximum growth rate is

\[
\text{Im}[\omega]_{\text{mcut}} = -\frac{1}{12} \frac{\bar{E}(\epsilon^2 G^2)}{12} + \frac{\bar{m}^2}{\bar{E}}(\epsilon^2 G^2) + \frac{\bar{m} G^2}{\bar{E}}(\epsilon^2 G^2) - \bar{E}. \tag{84}
\]

The maximum growth rate increases when the solid layer is deformable (\( H \neq 0 \)). These features are clearly seen in Fig. 3(a). In Fig. 3(b), we see that when \( H > 1.35 \), the growth rate becomes unbounded at two wavenumbers due to a zero value in the denominator of Eq. (84). This singularity could be removed by considering inertial terms, but for values of \( H < 1.35 \) the inertialess theory is expected to yield accurate results [11].

The growth rate can be examined in the limit of small \( H \) by performing a Taylor series expansion of Eq. (80):

\[
\text{Im}[\omega]_{\text{asymp}} = -k^2 \frac{1}{3} (k^2 S^a - \epsilon G)
+ k^2 \frac{H}{3E} k^2 (k^2 S^a - \epsilon G)^2 + O(H^2). \tag{85}
\]
found good agreement even when $H = 1$ provided that $G$ is sufficiently small. This can be rationalized by noting that the entire $O(H)$ term in Eq. (86) can be small even when $H = 1$ provided that the other parameters have suitable values.

Finally, it worthwhile to consider the magnitude of the effect predicted above. If we take $\eta = 10^{-3}$ Pa s, $\sigma = 0.01$ N/m, $\rho_l = 10^3$ kg/m$^3$, $E = 100$ Pa, and $\epsilon = 0.1$, then for $G = 10$ and $S_s = 1$ we find $R \sim 100 \mu m$ and $\dot{E} \sim 10^5$. The large value of $\dot{E}$ indicates that even for very soft solids ($\dot{E} \sim 100$ Pa), the enhancement of the growth rate due to solid deformability is expected to be weak for cases of practical interest. Nevertheless, without carrying out the analysis here, it would not have been obvious to determine whether solid deformability enhances or delays the film instability, and the manner in which it does so [cf. (85)]. It is also interesting to note that much stronger effects of solid deformability on liquid behavior have been observed experimentally in cases where the liquid is flowing (e.g., shear flow past a gel) [12–14]. In these cases, solid deformability can introduce new instabilities as well as modify existing ones.

V. CONCLUSIONS

Systematic conversion of the equations and boundary conditions governing solid deformation reveals that the continuity-of-velocity boundary condition at the liquid-solid interface is more complicated than has previously been assumed, even in the small-strain limit. Terms involving products of a displacement gradient and a time derivative appear and cannot be neglected in the small-strain limit unless the characteristic time scale is $O(L/U)$, where $L$ and $U$ are a characteristic length and velocity, respectively, in the lateral direction. The approach taken here thus makes clear the conditions under which the commonly used simplified version of the continuity-of-velocity boundary condition is valid.

The small-strain approximation, lubrication theory, and linear stability analysis are then applied to study the gravity-driven instability of a liquid film underneath a soft solid. Asymptotic analysis reveals that the coupling between the liquid and solid manifests itself as a lower effective liquid-air interfacial tension that leads to larger instability growth rates. Although this effect is expected to be weak for cases of practical interest, our work is limited to the linear regime and much stronger effects may take place in the nonlinear regime. The systematic approach taken here provides a framework that could be extended to study nonlinear effects, e.g., through the development of long-wave evolution equations [7]. Such studies will also require accounting for nonlinear constitutive behavior when the deformation gradients are no longer small.

ACKNOWLEDGMENTS

The authors acknowledge support from the Institute for Mathematics and its Applications at the University of Minnesota. S.K. also acknowledges support from the U.S. Department of Energy under Award No. DE-FG02-07ER46415.