Long Division of Polynomials

A rational expression is an algebraic fraction where both the numerator and denominator are polynomials. As always, the denominator cannot be zero. This fraction suggests ***division*** of a polynomial by another polynomial for which there are two methods:

A. Long Division
B. Synthetic Division

A. Long Division

I. **Long division** mimics the long division algorithm for real numbers. An example is presented below to review the steps and the vocabulary necessary to discuss the process of division. The 5-step division algorithm follows these general steps:

(1) Divide
(2) Multiply
(3) Subtract
(4) "Bring Down"
(5) Repeat or END

__Example:__ $\frac{762}{35}$ can be represent division of 762 by 35, i.e., $762 \div 35$.

$$\begin{array}{r} 21 \\ 35\overline{)762} \\ -70\downarrow \\ \hline 62 \\ -35 \\ \hline 27 \end{array}$$

$\therefore 762 \div 35 = 21 R 27$

(1) **Divide:** 35 "goes into" 76 two times
(2) **Multiply:** 2 times 35 = 70
(3) **Subtract:** 76 – 70 = 6
(4) **"Bring Down"** the 2 from the dividend
(5) **Repeat:**
  i) **Divide:** 35 "goes into" 62 one time
  ii) **Multiply:** 1 times 35 = 35
  iii) **Subtract:** 62 - 35 = 27
  iv) **Repeat or END:** Since there are no more digits to "bring down", the algorithm **ENDs**

Answer: $762 \div 35 = 21\frac{27}{35}$

or

$762 = (35)\cdot(21) + 27$ (Each part has a name)

Dividend = Divisor · Quotient + Remainder

We are now ready to mimic this procedure in the division of a polynomial by another polynomial.

II. Long Division of Polynomials

Division of a polynomial by another polynomial utilizes the same five algorithm steps as long division: (1) divide, (2) multiply, (3) subtract, (4) "bring down", (5) repeat or END.

Example: Divide ($x^4 - 9$) by ($x^2 + 2x + 3$).

A preliminary step needed for this example is to ensure all powers of the numerator (or dividend) are present so that a term for each power of x can be available as a placeholder. Thus we extend the dividend as below:

$$(x^4 + 0x^3 + 0x^2 + 0x - 9) \div (x^2 + 2x + 3)$$

(1) **Divide:** Polynomial division focuses **only on the leading terms**, hence $x^4 \div x^2 = x^2$
(2) **Multiply:** x^2 times $x^2 + 2x + 3 = x^4 + 2x^3 + 3x^2$
(3) **Subtract:** $(x^4 + 0x^3 + 0x^2) - (x^4 + 2x^3 + 3x^2) = -2x^3 - 3x^2$
(4) **"Bring Down":** $0x - 9$
(5) **Repeat** (the problem is now reduced to $(-2x^3 - 3x^2 + 0x - 9) \div (x^2 + 2x + 3)$
   a. **Divide:** $-2x^3 \div x^2 = -2x$
   b. **Multiply:** $-2x$ times $x^2 + 2x + 3 = -2x^3 - 4x^2 - 6x$
   c. **Subtract:** $(-2x^3 - 3x^2 + 0x - 9) - (-2x^3 - 4x^2 - 6x) = x^2 + 6x$
   d. **"Bring Down":** -9
   e. **Repeat:** (the problem is now reduced to $(x^2 + 6x - 9) \div (x^2 + 2x + 3)$
      i. **Divide:** $x^2 \div x^2 = 1$
      ii. **Multiply:** 1 times $x^2 + 2x + 3 = x^2 + 2x + 3$
      iii. **Subtract:** $(x^2 + 6x - 9) - (x^2 + 2x + 3) = 4x - 12$
      iv. **"Bring Down":** Nothing more to bring down (therefore we will end)
      v. **Repeat or End:** The quotient of dividing ($x^4 - 9$) by ($x^2 + 2x + 3$) is

$$\boldsymbol{x^2 - 2x + 1 + \frac{4x-12}{x^2+2x+3}}.$$

[See next page for a presentation of this process.]

Now you try the following problems:

1. $(5x^3 - 9x^2 + 24x - 5) \div (5x - 4)$
2. $(3x^3 - 7x + 8) \div (x - 5)$
3. $(6x^4 - 2x^3 - 3x + 2) \div (x^2 + 1)$

$$x^2-2x+1$$

$$x^2+2x+3\,\overline{)\,x^4+0x^3+0x^2+0x-9}$$

$$-\left(x^4+2x^3+3x^2\right)\ \downarrow$$

$$-2x^3-3x^2+0x-9$$

$$-\left(-2x^3-4x^2-6x\right)\ \downarrow$$

$$x^2+6x-9$$

$$-\left(x^2+2x+3\right)$$

$$4x-12$$ ← Remainder

$$\therefore\ (x^4-9)\div(x^2+2x+3)=x^2-2x+1+\frac{4x-12}{x^2+2x+3}$$ ← Quotient

B. Synthetic Division of Polynomial

Synthetic division is a simplified method that can be used to divide polynomials as long as the divisor is of the form $x - c$, where c is a constant. It is a method that occludes the variable and leaves just a skeletal form of the essential arithmetic manipulations. The following reflects how synthetic division (on right) relates to the formal polynomial division just explained.

$$1x^2 - 5x + 6$$

$$x + 1 \,\big)\, 1x^3 - 4x^2 + 1x + 7$$

$$\underline{-x^3 - 1x^2}$$

$$-5x^2 + x$$

$$\underline{5x^2 + 5x}$$

$$6x + 7$$

$$\underline{-6x - 6}$$

$$1$$

Remainder

c = -1, from the polynomial divisor $x + 1$ $= x - (-1)$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 7 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & 1 \end{array}$$

The formatting differs, but the essential arithmetical manipulation are observed to be the same in each.

Let's examine the synthetic division method used on the right.

1. **The Set-Up:** c | coefficients for each term of the polynomial (including 0 for those missing)

   Leave some workspace here.

   Bring down the leading coefficient.

   Our example yields the following set up:

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 7 \\ & & & & \\ \hline & 1 & & & \end{array}$$

2. Multiply the red 1 by the purple -1 and place product under the green -4.

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 7 \\ & & -1 & & \\ \hline & 1 & & & \end{array}$$

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3. Sum the newly formed column (-4 + -1 = -5) and place the sum below.

-1	1	-4	1	7
		-1		
	1	-5		

4. **Repeat:**
   a. Multiply the red -5 by the purple -1 and place sum under the green 1 and add.
   b. Multiply the red 6 by the purple -1 and place sum under the green 5 and add.
   c. Having reached the rightmost column, the process concludes.

a.

-1	1	-4	1	7
		-1	5	
	1	-5	6	

b.

-1	1	-4	1	7
		-1	5	-6
	1	-5	6	1

5. **Interpretation:** The results (at bottom, in red) are decoded as the polynomial quotient of degree one less than the original polynomial being divided. The last digit (here highlighted in yellow) represents the remainder. Hence, the number sequence 1, -5, 6, 1 is interpreted as $1x^2 - 5x + 6$, with a remainder of 1 (though, of course, the leading coefficient 1 would not be explicitly written out). And the final answer is therefore $x^2 - 5x + 6 + \frac{1}{x+1}$.

Now you divide the following problems ***using synthetic division***:

4. $(5x^3 + 2x^2 - x + 9) \div (x + 2)$
5. $(3x^4 - 7x + 8) \div (x - 1)$
6. $(x^3 + 8) \div (x + 2)$
7. $(2x^4 + x^3 - 2x^2 + 3x - 4) \div (x - 1)$

PART A Exercises KEY

① $(5x^3 - 9y^2 + 24x - 5) \div (5x - 4)$

$$\begin{array}{r|l} & x^2 - x + 4 \\ \hline 5x - 4 & 5x^3 - 9x^2 + 24x - 5 \\ & -5x^3 + 4x^2 \\ \hline & -5x^2 + 24x - 5 \\ & +5x^2 - 4x \\ \hline & 20x - 5 \\ & -20x + 16 \\ \hline & 11 \end{array}$$

$$\boxed{\therefore (5x^3 - 9x^2 + 24x - 5) \div (5x - 4) = x^2 - x + 4 + \frac{11}{5x - 4}}$$

② $(3x^3 - 7x + 8) \div (x - 5)$

$$\begin{array}{r|l} & 3x^2 + 15x + 68 \\ \hline x - 5 & 3x^3 + 0x^2 - 7x + 8 \\ & -3x^3 + 15x^2 \\ \hline & 15x^2 - 7x + 8 \\ & -15x^2 + 75x \\ \hline & 68x + 8 \\ & -68x + 340 \\ \hline & 348 \end{array}$$

$$\boxed{\therefore (3x^3 - 7x + 8) \div (x - 5) = 3x^2 + 15x + 68 + \frac{348}{x - 5}}$$

③ $(6x^4 - 2x^3 - 3x + 2) \div (x^2 + 1)$

$$
\begin{array}{r|l}
 & 6x^2 - 2x - 6 \\
\hline
x^2 + 1 & 6x^4 - 2x^3 + 0x^2 - 3x + 2 \\
 & -6x^4 + 0x^3 - 6x^2 \\
\cline{2-2}
 & -2x^3 - 6x^2 - 3x + 2 \\
 & +2x^3 + 0x^2 + 2x \\
\cline{2-2}
 & -6x^2 - x + 2 \\
 & +6x^2 + 0x + 6 \\
\cline{2-2}
 & -x + 8
\end{array}
$$

$$\boxed{\therefore (6x^4 - 2x^3 - 3x + 2) \div (x^2 + 1) = 6x^2 - 2x - 6 + \frac{8 - x}{x^2 + 1}}$$

PART B Exercises **KEY**

4.

$$\begin{array}{r|rrrr} -2 & 5 & 2 & -1 & 9 \\ & & -10 & 16 & -30 \\ \hline & 5 & -8 & 15 & -21 \end{array} \leftarrow \text{Remainder}$$

$$\therefore (5x^3+2x^2-x+9) \div (x+2)$$
$$= \boxed{5x^2-8x+15+\frac{-21}{x+2}}$$

5.

$$\begin{array}{r|rrrrr} 1 & 3 & 0 & 0 & -7 & 8 \\ & & 3 & 3 & 3 & -4 \\ \hline & 3 & 3 & 3 & -4 & 4 \end{array} \leftarrow \text{Remainder}$$

$$\therefore (3x^4-7x+8) \div (x-1)$$
$$= \boxed{3x^3+3x^2+3x-4+\frac{4}{x-1}}$$

6.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array} \leftarrow \text{No Remainder}$$

$$\therefore (x^3+8) \div (x+2) = \boxed{x^2-2x+4}$$

7.

$$\begin{array}{r|rrrrr} 1 & 2 & 1 & -2 & 3 & -4 \\ & & 2 & 3 & 1 & 4 \\ \hline & 2 & 3 & 1 & 4 & 0 \end{array} \leftarrow \text{No Remainder}$$

$$\therefore (2x^4+x^3-2x^2+3x-4) \div (x-1)$$
$$= \boxed{2x^3+3x^2+x+4}$$