Integrals Involving Products of Trig Functions

Three special cases where trigonometric substitutions can be utilized to evaluate an integral:

Case #1

$$\int \sin^n x \cos^m x dx$$

(with n or m odd)

a. If m is odd: make the substitution u= $\sin x$. Then du= $\cos x dx$. This uses one factor of $\cos x$. The remaining <u>even</u> factors of $\cos x$ can be converted to a function of $\sin x$ by the identity: $\cos^2 x = 1 - \sin^2 x$. The integral then has the form:

$$\int \sin^n x(1-\sin^2 x)^k \cos x dx = \int u^n (1-u^2)^k du$$

b. If n is odd: make the substitution u= $\cos x$. Then du= $-\sin x dx$. This uses one factor of $\sin x$. The remaining <u>even</u> factors of $\sin x$ can be converted to a function of $\cos x$ by the identity: $\sin^2 x = 1 - \cos^2 x$. The integral then has the form:

$$-\int \cos^m x(1-\cos^2 x)^k \sin x dx = -\int u^m (1-u^2)^k du$$

Case #2

$$\int \sin^n x \cos^m x dx$$

(with <u>both</u> n and m <u>even</u>)

Use the following identities repeatedly until an integrand involving only constants and cosine terms is obtained.

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

Case #3

$$\int sec^n x\ tan^m x dx$$

b. If n is even write $\sec^{n-2} x$ as a function of $\tan x$ using the identity $\sec^2 x = \tan^2 x + 1$. Then make the substitution u= $\tan x$. The remaining factor $\sec^2 x$ becomes du.

c. If m is odd write $\tan^{m-2} x$ as a function of $\sec x$ using the identity $\tan^2 x = \sec^2 x - 1$. Make the substitution u= $\sec x$ using one factor of $\sec x$ and using the remaining factor of $\tan x$ as du= $\sec x \tan x dx$.

d. If n=0, write

$$\int \tan^m x dx = \int \tan^{m-2} x(\tan^2 x)dx = \int \tan^{m-2} x(\sec^2 x - 1)dx =$$

$$\int \tan^{m-2} x(\sec^2 x)dx - \int \tan^{m-2} x dx$$

Integrate the first integral and repeat above process for $\int \tan^{m-2} x dx$

e. In none of the above cases apply; try rewriting the integrand in terms of sines and cosines or use integration by parts.

NOTE: For integrals involving powers of the cotangent and cosecant, follow the strategies of step 3 making use of the identity $\csc^2 x = 1 + \cot^2 x$

EXAMPLES:

1. Evaluate: $\int \sin^2 x \cos^5 x dx$

Solution:

Since m is odd, let u= $\sin x$. Then du= $\cos x dx$.

Write:

$$\int \sin^2 x \cos^5 x dx = \int \sin^2 x \cos^4 x(\cos x)dx =$$

$$= \int \sin^2 x(\cos^2 x)^2(\cos x)dx = \int \sin^2 x(1 - \sin^2 x)^2(\cos x)dx =$$

$$= \int u^2(1 - u^2)^2 du = \int u^2(1 - 2u^2 + u^4)du = \int (u^2 - 2u^4 + u^6)du =$$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C = \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

2. Evaluate: $\int \cos^4 x dx$

Solution:

Since m and n are both even (with n=0), replace $\cos^4 x$ by $\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)^2$

Write:

$$\int \cos^4 x dx = \int\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)^2 dx = \int\left(\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{\cos^2 2x}{4}\right)dx$$

Replace $\cos^2 2x$ by $\left(\frac{1}{2}+\frac{1}{2}\cos 4x\right)$

$$=\int\left[\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos 4x\right)\right]dx = \int\left(\frac{1}{4}+\frac{\cos 2x}{2}+\frac{1}{8}+\frac{1}{8}\cos 4x\right)dx =$$

$$=\int\left(\frac{3}{8}+\frac{\cos 2x}{2}+\frac{\cos 4x}{8}\right)dx = \frac{3}{8}\int dx+\frac{1}{4}\int(\cos 2x)(2)dx+\frac{1}{32}\int(\cos 4x)(4)dx =$$

$$=\frac{3}{8}x+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C$$

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3. Evaluate: $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}dx$

Solution:

Since m is odd, use the identity $\tan^2 x = \sec^2 x - 1$

Write:

$$\int \frac{\tan^3 x}{\sqrt{\sec x}} xdx = \int \sec^{-1/2} x \tan^3 x dx = \int \sec^{-1/2} x(\tan^2 x)(\tan x)dx$$

Replace $\sec^{-1/2} x$ by $\sec^{-3/2} x \sec x$

$$\int \sec^{-3/2} x \sec x(\tan^2 x)(\tan x)dx = \int \sec^{-3/2} x(\tan^2 x)(\tan x \sec x)dx =$$

$$=\int\left(\sec^{-3/2} x\right)(\sec^2 x - 1)(\tan x \sec x)dx$$

Let u= $\sec x$ and du= $\tan x \sec x dx$

$$=\int u^{-3/2}(u^2-1)du = \int\left(u^{1/2}-u^{-3/2}\right)du = \frac{u^{3/2}}{3/2}-\frac{u^{-1/2}}{-1/2}+C = \frac{2}{3}u^{3/2}+2u^{-1/2}+C =$$

$$=\frac{2(\sec x)^{3/2}}{3}+\frac{2}{(\sec x)^{1/2}}+C = \frac{2\sec^{3/2} x}{3}+\frac{2}{\sec^{1/2} x}+C$$

PRACTICE:

1. $\int \cos^2 x \tan^3 x dx$
2. $\int \tan^2 x dx$
3. $\int \tan^4 x \sec^6 x dx$
4. $\int \frac{\cos^5 \alpha}{\sqrt{\sin \alpha}} d\alpha$
5. $\int \frac{\cos x + \sin 2x}{\sin x} dx$
6. $\int \tan^3 x \sec x dx$

ANSWERS

1. $\frac{1}{2}\cos^2 x - \ln|\cos x| + C$

2. $\tan x - x + C$

3. $\frac{1}{9}\tan^9 x + \frac{2}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

4. $2\sin^{1/2}\alpha - \frac{4}{5}\sin^{5/2}\alpha + \frac{2}{9}\sin^{9/2}\alpha + C$

5. $\ln|\sin x| + 2\sin x + C$

6. $\frac{1}{3}\sec^3 x - \sec x + C$