Integrals Involving Products of Trig Functions

Three special cases where trigonometric substitutions can be utilized to evaluate an integral:

Case #1
\[ \int \sin^n x \cos^m x \, dx \]

(with n or m odd)

a. If \( m \) is odd: make the substitution \( u = \sin x \). Then \( du = \cos x \, dx \). This uses one factor of \( \cos x \). The remaining even factors of \( \cos x \) can be converted to a function of \( \sin x \) by the identity: \( \cos^2 x = 1 - \sin^2 x \). The integral then has the form:

\[ \int \sin^n x (1 - \sin^2 x)^k \, \cos x \, dx = \int u^n (1 - u^2)^k \, du \]

b. If \( n \) is odd: make the substitution \( u = \cos x \). Then \( du = -\sin x \, dx \). This uses one factor of \( \sin x \). The remaining even factors of \( \sin x \) can be converted to a function of \( \cos x \) by the identity: \( \sin^2 x = 1 - \cos^2 x \). The integral then has the form:

\[ -\int \cos^m x (1 - \cos^2 x)^k \, \sin x \, dx = -\int u^m (1 - u^2)^k \, du \]

Case #2
\[ \int \sin^n x \cos^m x \, dx \]

(with both \( n \) and \( m \) even)

Use the following identities repeatedly until an integrand involving only constants and cosine terms is obtained.

\[ \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x \quad \text{and} \quad \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \]
Case #3
\[ \int sec^n x \tan^m x dx \]

b. If n is even write \( sec^{n-2}x \) as a function of \( \tan x \) using the identity 
\( \sec^2 x = \tan^2 x + 1 \). Then make the substitution \( u = \tan x \). The remaining factor 
\( \sec^2 x \) becomes \( du \).
c. If m is odd write \( \tan^{m-2} x \) as a function of \( \sec x \) using the identity 
\( \tan^2 x = \sec^2 x - 1 \). Make the substitution \( u = \sec x \) using one factor of \( \sec x \) and using the remaining 
factor of \( \tan x \) as \( du = \sec x \tan x dx \).
d. If \( n=0 \), write 
\[ \int tan^m x dx = \int tan^{m-2} x(tan^2 x)dx = \int tan^{m-2} x(sec^2 x - 1)dx = \]
\[ \int tan^{m-2} x(sec^2 x)dx - \int tan^{m-2} x dx \]

Integrate the first integral and repeat above process for \( \int tan^{m-2} x dx \)
e. In none of the above cases apply; try rewriting the integrand in terms of sines and 
cosines or use integration by parts.

NOTE: For integrals involving powers of the cotangent and cosecant, follow the strategies of 
step 3 making use of the identity \( \csc^2 x = 1 + \cot^2 x \)

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**EXAMPLES:**

1. Evaluate: \( \int \sin^2 x \cos^5 x dx \)

Solution:

Since \( m \) is odd, let \( u = \sin x \). Then \( du = \cos x dx \).

Write:

\[ \int \sin^2 x \cos^5 x dx = \int \sin^2 x \cos^4 x (\cos x) dx = \]
\[ = \int \sin^2 x (\cos^2 x)^2 (\cos x) dx = \int \sin^2 x (1 - \sin^2 x)^2 (\cos x) dx = \]
\[ = \int u^2 (1 - u^2)^2 du = \int u^2 (1 - 2u^2 + u^4) du = \int (u^2 - 2u^4 + u^6) du = \]
\[ = \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C = \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C \]

2. Evaluate: \( \int \cos^4 x dx \)
Solution:

Since m and n are both even (with n=0), replace \( \cos^4 x \) by \( \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right)^2 \)

Write:

\[
\int \cos^4 x \, dx = \int \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right)^2 \, dx = \int \left( \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{\cos^2 2x}{4} \right) \, dx
\]

Replace \( \cos^2 2x \) by \( \left( \frac{1}{2} + \frac{1}{2} \cos 4x \right) \)

\[
\begin{align*}
= \int \left[ \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \left( \frac{1}{2} + \frac{1}{2} \cos 4x \right) \right] \, dx = \int \left( \frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{8} + \frac{1}{8} \cos 4x \right) \, dx = \\
= \int \left( \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8} \right) \, dx = \frac{3}{8} \int dx + \frac{1}{4} \int (\cos 2x)(2) \, dx + \frac{1}{32} \int (\cos 4x)(4) \, dx = \\
= \frac{3}{8} x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C
\end{align*}
\]

3. Evaluate: \( \int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx \)

Solution:

Since m is odd, use the identity \( \tan^2 x = \sec^2 x - 1 \)

Write:

\[
\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx = \int \sec^{\frac{1}{2}} x \tan^3 x \, dx = \int \sec^{\frac{1}{2}} x (\tan^2 x) (\tan x) \, dx
\]

Replace \( \sec^{\frac{1}{2}} x \) by \( \sec^{\frac{1}{2}} x \sec x \)

\[
\begin{align*}
\int \sec^{\frac{1}{2}} x \sec x (\tan^2 x) (\tan x) \, dx &= \int \sec^{\frac{1}{2}} x (\tan^2 x) (\tan x \sec x) \, dx = \\
&= \int \left( \sec^{-\frac{1}{2}} x \right) (\sec^2 x - 1) (\tan x \sec x) \, dx
\end{align*}
\]

Let \( u = \sec x \) and \( du = \tan x \sec x \, dx \)

\[
\begin{align*}
&= \int u^{-\frac{1}{2}} (u^2 - 1) \, du = \int \left( u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) \, du = \left. \frac{u^{\frac{3}{2}}}{3/2} - \frac{u^{-\frac{1}{2}}}{-1/2} \right| = \frac{2}{3} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C = \\
&= \frac{2}{3} (\sec x)^{\frac{3}{2}} + 2(\sec x)^{\frac{1}{2}} + C = \frac{2}{3} \sec^{\frac{3}{2}} x + \frac{2}{\sec^{\frac{1}{2}} x} + C
\end{align*}
\]

PRACTICE:
1. \( \int \cos^2 x \tan^3 x \, dx \)
2. \( \int \tan^2 x \, dx \)
3. \( \int \tan^4 x \sec^6 x \, dx \)
4. \( \int \frac{\cos^5 \alpha}{\sqrt{\sin \alpha}} \, d\alpha \)
5. \( \int \frac{\cos x + \sin 2x}{\sin x} \, dx \)
6. \( \int \tan^3 x \sec x \, dx \)
1. \( \frac{1}{2} \cos^2 x - \ln|\cos x| + C \)

2. \( \tan x - x + C \)

3. \( \frac{1}{9} \tan^9 x + \frac{2}{7} \tan^7 x + \frac{1}{5} \tan^5 x + C \)

4. \( 2 \sin^{\frac{1}{2}} \alpha - \frac{4}{5} \sin^{\frac{5}{2}} \alpha + \frac{2}{9} \sin^{\frac{9}{2}} \alpha + C \)

5. \( \ln|\sin x| + 2 \sin x + C \)

6. \( \frac{1}{3} \sec^3 x - \sec x + C \)