Integration by Trig Substitution

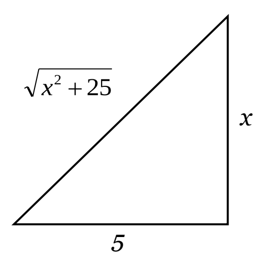
Outline of Procedure:

1) Construct a right triangle, fitting to the legs and hypotenuse that part of the integral that is, or resembles, the Pythagorean Theorem.
2) Using the triangle built in (1), form the various terms appearing in the integral in terms of trig functions. **Be sure to express *dx* in terms of a trig function also.**
3) Combine the trig functions from (2) to yield a trigonometrically equivalent expression of the original integral.
4) Use trig identities and/or algebra to simplify the expression developed in (3).
5) Integrate the expression developed in (4).
6) Use the triangle [step (1)], trig identities, and algebra (any combination of these, as needed) to express the answer to the integration [step (5)] in terms of *x*. That is, transform out of the trig functions and back to algebraic expressions.

<u>EXAMPLES</u>

A. $\int \frac{1}{x^2+25}dx$

1) x^2+25 resembles the Pythagorean Theorem where x and 5 are the legs. With this choice, the hypotenuse would be $\sqrt{x^2+25}$.



2) From the triangle,

$$\cos\theta = \frac{5}{\sqrt{x^2+25}}, \text{ so } \frac{1}{5}\cos\theta = \frac{1}{\sqrt{x^2+25}} \text{ and } \frac{1}{25}\cos^2\theta = \frac{1}{x^2+25}$$

also, $\tan\theta = \frac{x}{5}$, so $5\tan\theta = x$ and $5sec^2\theta\, d\theta = dx$.

3) $\frac{1}{x^2+25}dx = \left(\frac{1}{25}\cos^2\theta\right)(5\sec^2\theta\, d\theta)$

4) $\left(\frac{1}{25}\cos^2\theta\right)(5\sec^2\theta\, d\theta) = \frac{1}{5}d\theta$

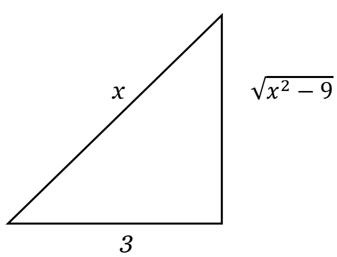
5) $\int \frac{1}{5}d\theta = \frac{1}{5}\theta$

6) From the triangle and step (2), $\tan\theta = \frac{x}{5}$ so $\theta = \tan^{-1}\frac{x}{5}$.

$\therefore \frac{1}{5}\theta = \frac{1}{5}\tan^{-1}\frac{x}{5}$, that is, $\int \frac{1}{x^2+25}dx = \frac{1}{5}\tan^{-1}\frac{x}{5} + C$

B. $\int \frac{\sqrt{x^2-9}}{x^2}dx$

1) $\sqrt{x^2-9}$ is the Pythagorean Theorem with x on the hypotenuse and 3 on a leg. With this choice, the other leg becomes $\sqrt{x^2-9}$.



2) From the triangle, $\tan\theta = \frac{\sqrt{x^2-9}}{3}$ so $3\tan\theta = \sqrt{x^2-9}$.

Also, $\cos\theta = \frac{3}{x}$ so $\frac{1}{3}\cos\theta = \frac{1}{x}$ and $\frac{1}{9}\cos^2\theta = \frac{1}{x^2}$.

Finally, $\sec\theta = \frac{x}{3}$ so $3\sec\theta\tan\theta\, d\theta = dx$

3) $\frac{\sqrt{x^2-9}}{x^2}dx = (3\tan\theta)\left(\frac{1}{9}\cos^2\theta\right)(3\sec\theta\tan\theta\, d\theta)$

4) $(3\tan\theta)\left(\frac{1}{9}\cos^2\theta\right)(3\sec\theta\tan\theta\, d\theta) = \left(\frac{\sin\theta}{\cos\theta}\right)(\cos^2\theta)\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)d\theta = \frac{\sin^2\theta}{\cos\theta}d\theta$

5) $\int\frac{\sin^2\theta}{\cos\theta}d\theta = \int\frac{1-\cos^2\theta}{\cos\theta}d\theta = \int\frac{1}{\cos\theta}d\theta - \int\frac{\cos^2\theta}{\cos\theta}d\theta = \int\sec\theta\, d\theta - \int\cos\theta\, d\theta,$

which equals $ln|\sec\theta + \tan\theta| - \sin\theta.$

6) From the triangle [step (1)] and/or step (2)]:

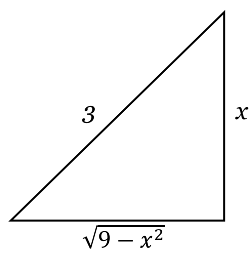
$\sec\theta = \frac{x}{3},\quad \tan\theta = \frac{\sqrt{x^2-9}}{x^2}, and\ \sin\theta = \frac{\sqrt{x^2-9}}{x},$

so, $\ln|\sec\theta + \tan\theta| - \sin\theta = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x}$

$\therefore \int\frac{\sqrt{x^2-9}}{x^2}dx = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x} + C$

C. $\int\frac{x^2}{\sqrt{9-x^2}}dx$

1) $\sqrt{9-x^2}$ is the Pythagorean theorem with 3 on the hypotenuse and x on one of the legs. With this choice, the other leg is $\sqrt{9-x^2}$.



2) From the triangle,

$\sec\theta = \frac{3}{\sqrt{9-x^2}},\quad$ so $\quad\frac{1}{3}\sec\theta = \frac{1}{\sqrt{9-x^2}}.$

Also, $\quad\sin\theta = \frac{x}{3}\quad$ so $\quad 3\sin\theta = x\quad and\quad 9\sin^2\theta = x^2$

Finally, since $3\sin\theta = x,\quad 3\cos\theta\, d\theta = dx.$

3) $\frac{3}{\sqrt{9-x^2}}dx = \left(\frac{1}{3}\sec\theta\right)(9\sin^2\theta)(3\cos\theta\, d\theta)$

4) $\left(\frac{1}{3}\sec\theta\right)(9\sin^2\theta)(3\cos\theta\, d\theta) = 9\sin^2\theta\, d\theta$

5) $\int 9\sin^2\theta\, d\theta = 9\int\frac{1}{2}(1-\cos(2\theta)d\theta) = \frac{9}{2}[\int d\theta - \int\cos(2\theta)d\theta] = \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)$

6) The answer from step (5) involves a double angle [sin (2θ)] which conflicts with the triangle in step (1) because the triangle is constructed around the single angle, θ, not the double angle 2θ. But using the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ resolves this problem. Therefore, $\frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) = \frac{9}{2}(\theta - \sin(\theta)\cos(\theta))$. Now, from the triangle and/or step (2), $\sin\theta = \frac{x}{3}$ so, $\theta = \sin^{-1}\frac{x}{3}$. Also, $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ ∴ $\frac{9}{2}(\theta - \sin(\theta)\cos(\theta)) = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right] = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \left(\frac{x\sqrt{9-x^2}}{9}\right)\right]$.

That is, $\int\frac{x^2}{\sqrt{9-x^2}}dx = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \left(\frac{x\sqrt{9-x^2}}{9}\right)\right] + C$

PRACTICE PROBLEMS

1. $\int\frac{x}{\sqrt{9-x^2}}dx$
2. $\int\sqrt{16-x^2}\,dx$
3. $\int\frac{1}{\sqrt{4-x^2}}dx$
4. $\int\frac{x}{x^2+25}dx$
5. $\int\frac{1}{15-x^2}dx$
6. $\int\frac{1}{x\sqrt{16+x^2}}dx$
7. $\int\frac{1}{\sqrt{3-7x^2}}dx$

ANSWERS

1. $-\sqrt{9-x^2}+C$

2. $8\left[\sin^{-1}\left(\frac{x}{4}\right)+\frac{x\sqrt{16-x^2}}{16}\right]+C$

3. $\sin^{-1}\left(\frac{x}{2}\right)+C$

4. $\ln\left|\frac{\sqrt{x^2+25}}{5}\right|+C$

5. $\frac{1}{\sqrt{15}}\ln\left|\frac{\sqrt{15}+x}{\sqrt{15-x^2}}\right|+C$

6. $\frac{1}{4}\ln\left|\frac{\sqrt{16+x^2}-4}{x}\right|+C$

7. $\frac{1}{\sqrt{7}}\sin^{-1}\left(\frac{\sqrt{7}x}{\sqrt{3}}\right)+C$