



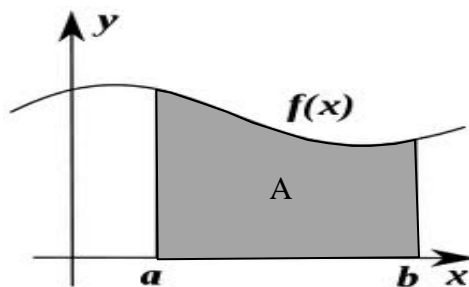
# Areas by Integration

1. Area under a curve – region bounded by the given function, vertical lines and the x –axis.
2. Area under a curve – region bounded by the given function, horizontal lines and the y –axis.
3. Area between curves defined by two given functions.

1. Area under a curve – region bounded by the given function, vertical lines and the x –axis.

If  $f(x)$  is a continuous and nonnegative function of  $x$  on the closed interval  $[a, b]$ , then the area of the region bounded by the graph of  $f$ , the  $x$ -axis and the vertical lines  $x=a$  and  $x=b$  is given by:

$$Area = \int_a^b f(x) dx$$

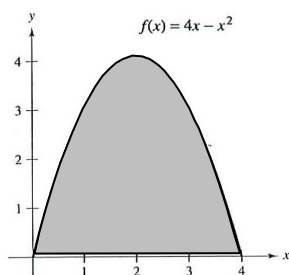


When calculating the area under a curve  $f(x)$ , follow the steps below:

1. Sketch the area.
2. Determine the boundaries  $a$  and  $b$ ,
3. Set up the definite integral,
4. Integrate.

**Ex. 1.** Find the area in the first quadrant bounded by  $f(x) = 4x - x^2$  and the  $x$ -axis.

Graph:



To find the boundaries, determine the  $x$ -intercepts:  $f(x) = 0 \rightarrow 4x - x^2 = 0$

$$x(4 - x) = 0$$

$$x = 0 \text{ or } (4 - x) = 0 \text{ so } x = 0 \text{ and } x = 4$$

Therefore the boundaries are  $a = 0$  and  $b = 4$

Set up the integral:  $A = \int_a^b f(x)dx = \int_0^4 (4x - x^2)dx$

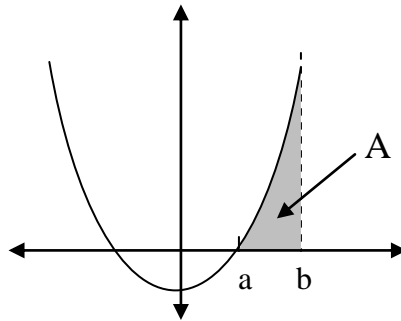
Solve:

$$\begin{aligned} \int_0^4 (4x - x^2)dx &= \left(4 \cdot \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\bigg|_0^4 = \left(2x^2 - \frac{1}{3}x^3\right)\bigg|_0^4 = \left(2 \cdot (4)^2 - \frac{1}{3}(4)^3\right) - \left(2 \cdot (0)^2 - \frac{1}{3}(0)^3\right) \\ &= \left(2 \cdot 16 - \frac{1}{3} \cdot 64\right) - 0 = \frac{32}{3} \end{aligned}$$

The area in the first quadrant under the curve  $f(x) = 4x - x^2$  is equal to  $\frac{32}{3}$  square units

**Ex. 2.** Find the area bounded by the following curves:  $y = x^2 - 4$ ,  $y = 0$ ,  $x = 4$ ,

Graph:



Finding the boundaries:

$$y = x^2 - 4, \text{ and } y = 0 \text{ implies } x^2 - 4 = 0 \text{ so } (x - 2)(x + 2) = 0$$

$$x = -2 \text{ or } x = 2$$

From the graph we see that  $x = 2$  is our boundary at a. The value  $x = -2$  is a solution to the equation above but it is not bounding the area. (Here's why the graph is an important tool to help us determine correct results. *Don't skip this step!*)

The other boundary value is given by the equation of the vertical line  $x = 4$ ,

Boundaries are:  $a = 2$ , and  $b = 4$ ,

Set up the integral:

$$A = \int_a^b f(x)dx = \int_2^4 (x^2 - 4)dx$$

Solve:

$$\begin{aligned} \int_2^4 (x^2 - 4)dx &= \left(\frac{1}{3}x^3 - 4x\right)\bigg|_2^4 = \left(\frac{1}{3} \cdot (4)^3 - 4 \cdot 4\right) - \left(\frac{1}{3} \cdot (2)^3 - 4 \cdot 2\right) \\ &= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) = \frac{64}{3} - 16 - \frac{8}{3} + 8 = \frac{56}{3} - 8 = \frac{32}{3} \end{aligned}$$

The area bounded by the curves  $y = x^2 - 4$ ,  $y = 0$ ,  $x = 4$ , is equal to  $\frac{32}{3}$  square units.

**2. Area under a curve –given function, region bounded by the horizontal lines and the y –axis.**

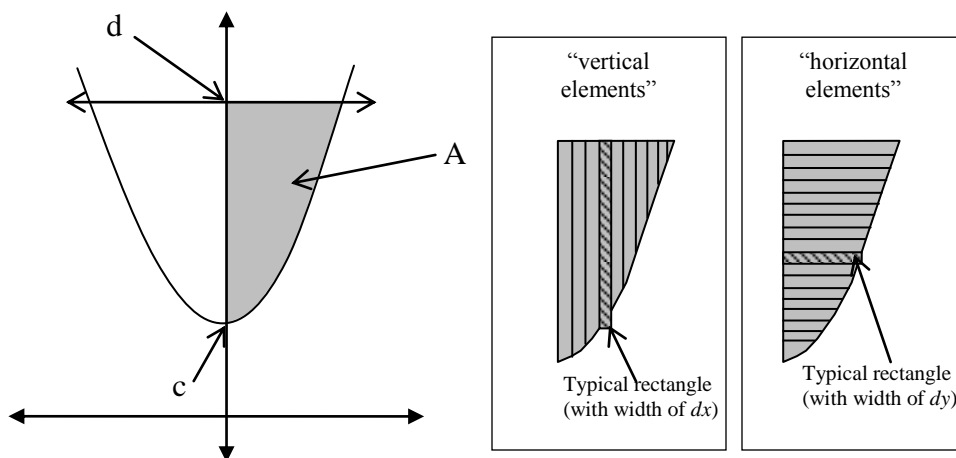
In certain problems it is easier to rewrite the function in terms of  $y$  and calculate the area using horizontal elements. In this case the formula for the area would be:

$$Area = \int_c^d g(y)dy$$

When calculating the area under a curve, or in this case to the left of the curve  $g(y)$ , follow the steps below:

1. Sketch the area.
2. Determine the boundaries  $c$  and  $d$ ,
3. Set up the definite integral,
4. Integrate.

**Ex. 3.** Find the first quadrant area bounded by the following curves:  $y = x^2 + 2$ ,  $y = 4$  and  $x = 0$ .



There are two ways to solve this problem: we can calculate the area between two functions  $y = 4$  and  $y = x^2 + 2$  using the vertical elements and integrate with respect to  $x$ , or we can use the horizontal elements and calculate the area between the  $y$ -axis and the function  $y = x^2 + 2$  integrating the functions with respect to  $y$ .

We will solve it using the second approach by considering horizontal elements and the function in terms of  $y$ .

The formula we will use is:  $Area = \int_c^d g(y)dy$ , so we need to determine the boundary values

$c$  and  $d$  first.

The boundary value  $d$  corresponds to the horizontal line  $y = 4$ . To calculate  $c$ , we need to locate the  $y$ -intercepts. Therefore set  $x = 0$  and solve the equation  $y = x^2 + 2$  for  $y$ ; hence  $y = 2$ . So the boundaries are  $c = 2$  and  $d = 4$ .

Now we need to find our  $g(y)$ . That is easily done by solving  $y = x^2 + 2$  for  $x$ .

$$y = x^2 + 2$$

$$x^2 = y - 2$$

$$x = \pm\sqrt{y - 2}$$

We will ignore the negative radical, since our area is in the first quadrant.

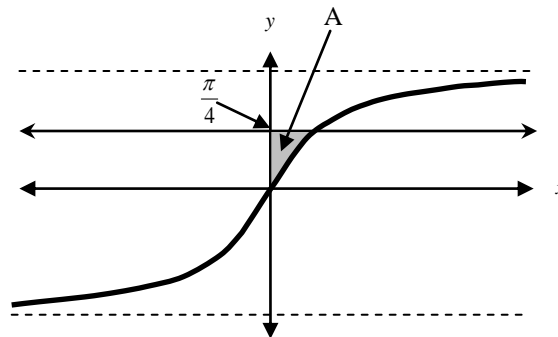
Now let's set up the integral:  $A = \int_2^4 \sqrt{y - 2} dy$

Solve the integral using a simple  $u$  substitution:

$$A = \int_2^4 \sqrt{y - 2} dy = \frac{2}{3} (\sqrt{y - 2})^3 \Big|_2^4 = \frac{2}{3} (\sqrt{4 - 2})^3 - \frac{2}{3} (\sqrt{2 - 2})^3 = \frac{2}{3} (\sqrt{2})^3 - 0 = \frac{4\sqrt{2}}{3} \text{ square units.}$$

The first quadrant area bounded by the following curves:  $y = x^2 + 2$ ,  $y = 4$  and  $x = 0$  is equal to  $\frac{4\sqrt{2}}{3}$  square units.

**Ex. 4.** Find the first quadrant area bounded by the following curves:  $y = \arctan x$ ,  $y = \frac{\pi}{4}$  and  $x = 0$ .



Since it is much easier to integrate  $x = \tan y$  than  $y = \arctan x$ , we will rewrite the given function in terms of  $y$ , and integrate using the horizontal elements and the formula:

$$A = \int_c^d g(y) dy \text{ to find the area. The function } y = \arctan x \text{ implies } x = \tan y. \text{ So } g(y) = \tan y.$$

The lower boundary  $c=0$  is easily obtained from the graph or by solving the equation  $\arctan x = 0$ . The upper boundary is given by the equation of the line:  $y = \frac{\pi}{4}$ .

So the area we are looking for is given by the following integral:  $A = \int_0^{\pi/4} \tan y dy$

Solving the integral yields:

$$\int_0^{\pi/4} \tan y dy = -\ln|\cos y| \Big|_0^{\pi/4} = -\ln\left|\cos \frac{\pi}{4}\right| - (-\ln|\cos 0|) = -\ln \frac{\sqrt{2}}{2} + \ln|1| =$$

$$\ln 2 - \ln \sqrt{2} + 0 = \ln 2 - \frac{1}{2} \ln 2 = \frac{1}{2} \ln 2$$

So the first quadrant area bounded by the following curves:  $y = \arctan x$ ,  $y = \frac{\pi}{4}$  and  $x = 0$  is equal to  $\frac{1}{2} \ln 2$  square units.

### 3. Area between two curves.

This can be considered as a more general approach to finding areas. Thus each of the previous examples could have been solved using such an approach by considering the  $x$ - and  $y$ - axes as functions with equations  $y=0$  and  $x=0$ , respectively.

Many areas can be viewed as being bounded by two or more curves. When area is enclosed by just two curves, it can be calculated using vertical elements by subtracting the lower function from the upper function and evaluating the integral.

Analogously, to calculate the area between two curves using horizontal elements, subtract the left function from the right function.

As always, a sketch of the graph can be a very important tool in determining the precise set-up of the integral. If you subtract in the wrong order, your result will be negative. That mistake can be avoided by taking the absolute value of the difference of the functions.

Here is the universal formula for finding the area between two curves:

- Using the vertical elements:

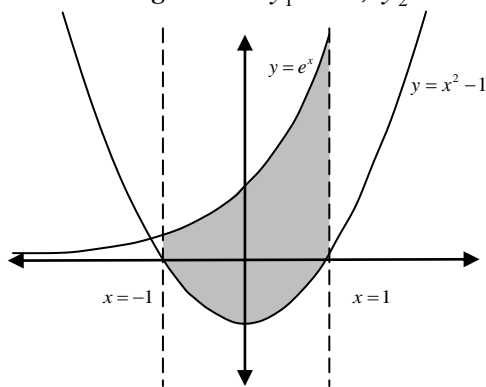
$$Area = \int_a^b |y_2 - y_1| dx \text{ where } y_1 \text{ and } y_2 \text{ are functions of } x$$

- Using the horizontal elements:

$$Area = \int_c^d |x_2 - x_1| dy \text{ where } x_1 \text{ and } x_2 \text{ are functions of } y.$$

**Ex.5.** Find the area of the region enclosed by the following curves:  $y_1 = e^x$ ,  $y_2 = x^2 - 1$ ,  $x = -1$  and  $x = 1$ .

As always, we will first draw a sketch.



In this case it is fairly easy to integrate the functions as given with respect to  $x$ . So the boundaries are:  $a = -1$  and  $b = 1$ .

Notice that in the region that we are interested in, the function  $y_1 = e^x$  is above the function  $y_2 = x^2 - 1$ , thus the integral set up should look as follows:

$$A = \int_{-1}^1 [e^x - (x^2 - 1)] dx$$

Solving it:

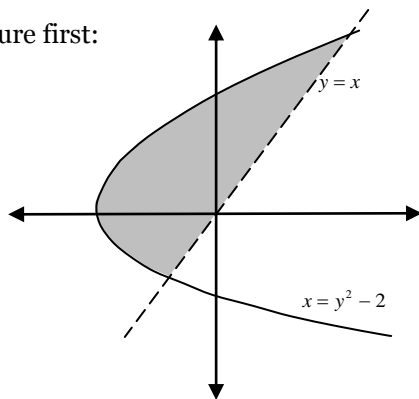
$$\begin{aligned} \int_{-1}^1 [e^x - (x^2 - 1)] dx &= \int_{-1}^1 (e^x - x^2 + 1) dx = e^x - \frac{1}{3}x^3 + x \Big|_{-1}^1 = \left( e^1 - \frac{1}{3}(1)^3 + 1 \right) - \left( e^{-1} - \frac{1}{3}(-1)^3 + (-1) \right) \\ &= \left( e + \frac{2}{3} \right) - \left( \frac{1}{e} - \frac{2}{3} \right) = e - \frac{1}{e} + \frac{4}{3} = \frac{e^2 - 1}{e} + \frac{4}{3} = \frac{3e^2 + 4e - 3}{3e}. \end{aligned}$$

So the area of the region bounded by  $y_1 = e^x$ ,  $y_2 = x^2 - 1$ ,  $x = -1$  and  $x = 1$  is equal to

$$\frac{3e^2 + 4e - 3}{3e} \text{ square units.}$$

**Ex.6.** Find the area of the region enclosed by the following curves:  $x_1 = y^2 - 2$ , and  $x_2 = y$ . Since the first function is better defined as a function of  $y$ , we will calculate the integral with respect to  $y$ .

As usual – draw the picture first:



In this case the boundaries are determined by the points of intersection of both functions. Remember that we want the  $y$ -values since we will be integrating with respect to  $y$ .

We need  $x_1 = x_2$ . This implies  $y^2 - 2 = y$  and

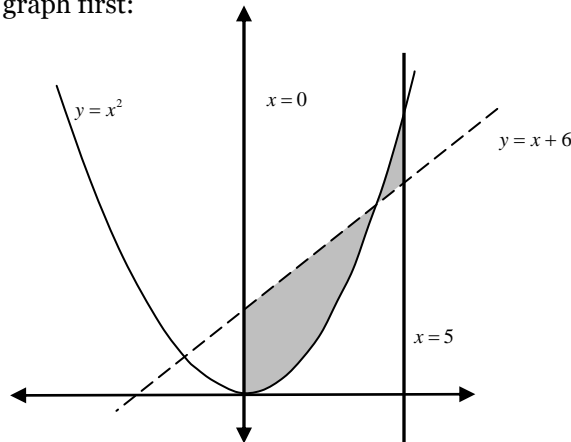
$$\begin{aligned} y^2 - y - 2 &= 0 \\ (y - 2)(y + 1) & \\ y &= 2 \text{ or } y = -1 \end{aligned}$$

So  $c = -1$  and  $d = 2$ . The “left” function is  $x_1 = y^2 - 2$  and the “right” function is  $x_2 = y$ .

$$\begin{aligned} A &= \int_{-1}^2 (y - (y^2 - 2)) dy = \frac{1}{2}y^2 - \frac{1}{3}y^3 + 2y \Big|_{-1}^2 = \left( \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 + 2 \cdot 2 \right) - \left( \frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3 + 2 \cdot (-1) \right) \\ &= \left( 2 - \frac{8}{3} + 4 \right) - \left( \frac{1}{2} + \frac{1}{3} - 2 \right) = 6 - \frac{8}{3} - \frac{1}{3} + 2 - \frac{1}{2} = 4\frac{1}{2} = \frac{9}{2} \text{ square units.} \end{aligned}$$

So the area of the region enclosed by the curves:  $x_1 = y^2 - 2$ , and  $x_2 = y$  is equal to  $\frac{9}{2}$  square units.

**Ex.7.** Find the area of the region enclosed by the following curves:  $y = x^2$ ,  $y = x + 6$ ,  $x = 0$  and  $x = 5$   
 As usual – sketch a rough graph first:



In this case it is very important to draw the graph, since the functions intersect between the boundaries. This means that we will have to actually calculate two separate integrals and then add the results. Otherwise we would end up subtracting the two pieces from each other.

First we need the “middle” intersection point so we will solve the equation:  $x^2 = x + 5$

$$\begin{aligned} x^2 - x - 5 &= 0 \\ (x - 3)(x + 2) &= 0 \\ x &= 3 \text{ or } x = -2 \end{aligned}$$

The intersection point at  $x = -2$  is outside our area. We are interested in  $x = 3$ , this is our “middle” boundary value.

In this case the integral set-up will look as follows:

$$\begin{aligned} A &= \int_0^3 (x + 6 - x^2) dx + \int_3^5 (x^2 - (x + 6)) dx = \left( \frac{1}{2} x^2 + 6x - \frac{1}{3} x^3 \Big|_0^3 \right) + \left( \frac{1}{3} x^3 - \frac{1}{2} x^2 - 6x \Big|_3^5 \right) = \\ &= \left( \frac{1}{2} \cdot 3^2 + 6 \cdot 3 - \frac{1}{3} \cdot 3^3 \right) + \left( \frac{1}{3} \cdot 5^3 - \frac{1}{2} \cdot 5^2 - 6 \cdot 5 \right) - \left( \frac{1}{3} \cdot 3^3 - \frac{1}{2} \cdot 3^2 - 6 \cdot 3 \right) = \\ &\frac{9}{2} + 18 - 9 + \frac{125}{3} - \frac{25}{2} - 30 - 9 + \frac{9}{2} + 18 = \frac{157}{6} \text{ square units.} \end{aligned}$$

So the area of the region enclosed by the curves:  $y = x^2$ ,  $y = x + 6$ ,  $x = 0$  and  $x = 5$

is equal to  $\frac{157}{6}$  square units.

### Practice problems:

Find the area of the region bounded by the given curves. Decide whether to integrate with respect to  $x$  or  $y$ .

1.  $y = 3x^2$ ,  $y = 0$ , and  $x = 3$

2.  $y = x^2$ ,  $y = 2 - x$ ,  $x = 0$  and  $x \geq 0$

3.  $y = e^x$ ,  $y = xe^x$ , and  $x = 0$

4.  $x = 2y^2$  and  $x = 4 + y^2$

5.  $y = \cos x$ ,  $y = \sin 2x$ ,  $x = 0$  and  $x = \frac{\pi}{2}$



**Answers:**

1.  $A = 27$
2.  $A = \frac{7}{6}$
3.  $A = e^{-2}$
4.  $A = \frac{32}{3}$
5.  $A = \frac{1}{2}$