Non-Homogeneous Second Order Differential Equations

Procedure for solving non-homogeneous second order differential equations:

$$y'' + p(x)y' + q(x)y = g(x)$$

1. Determine the general solution $y_h = C_1 y(x) + C_2 y(x)$ to a homogeneous second order differential equation: $y'' + p(x)y' + q(x)y = 0$
2. Find the particular solution y_p of the non-homogeneous equation, using one of the methods below.
3. The general solution of the non-homogeneous equation is: $y(x) = C_1 y(x) + C_2 y(x) + y_p$ where C_1 and C_2 are arbitrary constants.

METHODS FOR FINDING THE PARTICULAR SOLUTION (y_p) OF A NON-HOMOGENOUS EQUATION	
Undetermined Coefficients. Restrictions: 1. D.E must have constant coefficients: $ay'' + by' + c = g(x)$ 2. g(x) must be of a certain, "easy to guess" form.	1. Write down g(x). Start taking derivatives of g(x). List all the terms of g (x) and its derivatives while ignoring the coefficients. Keep taking the derivatives until no new terms are obtained. 2. Compare the listed terms to the terms of the homogeneous solution. If one or more terms are repeating, then the recurring expression needs to be modified by multiplying all the repeating terms by x. 3. Based on step 1 and 2 create an initial guess for y_p. 4. Take the 1st and the 2nd derivatives of y_p. Plug into the differential equation. Solve for the constants. 5. Plug the values of the constants into y_p.
Variation of Parameters.	$y_p(x) = -y_1 \int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}dx + y_2 \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}dx$ where y_1 and y_2 are solutions to the homogeneous equation and $W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$ The set of solutions is linearly independent in *I* if $W(y_1, y_2)(x) \neq 0$ for every x in the interval. **Or equivalently:** $y_p(x) = v_1 y_1 + v_2 y_2$ where y_1 and y_2 are solutions to the homogeneous equation and v_1 and v_2 are unknown functions of *x*. To determine v_1 and v_2, solve the following system of equations for v'_1 and v'_2. $y_1 v_1' + y_2 v_2' = 0$ $y_1' v_1' + y_2' v_2' = g(x)$ Integrate v'_1 and v'_2 to find v_1 and v_2. Substitute v_1 and v_2 into $y_p(x) = v_1 y_1 + v_2 y_2$

Example #1. Solve the differential equation: $y'' - 2y' = t + e^t$

Solution:

1. Homogeneous equation: $y'' - 2y' = 0$

   Characteristic equation: $r^2 - 2r = 0$

   $r(r-2) = 0$

   $r = 0, r = 2$

   $\Rightarrow y_h = C_1 + C_2 e^{2t}$

2. Particular solution:

   $g(t) = t + e^t$ — t, e^t

   $g'(t) = 1 + e^t$ $\Rightarrow$ Terms: C, e^t

   $g''(t) = e^t$ — e^t No new terms.

   The constant is already in the homogeneous solution. Multiplying it by t will repeat the terms of g(t). So we need to modify both the constant and the t.

Initial guess of y_p $\quad y_p = \underbrace{(At + B)} + Ce^t$

Part of a homogeneous solution. Both terms need to be modified

Modify y_p: $\quad y_p = t(At + B) + Ce^t = At^2 + Bt + Ce^t$

$$y_p' = 2At + B + Ce^t$$

$$y''_p = 2A + Ce^t$$

Plug the y_p and its derivatives into the original differential equation:

$y'' - 2y' = t + e^t$ implies $2A + Ce^t - 2(2At + B + Ce^t) = t + e^t$

$$2A - 2B - 4At - Ce^t = t + e^t \Rightarrow$$

$$-C = 1 \Rightarrow C = -1$$

$$-4A = 1 \Rightarrow A = -\frac{1}{4}$$

$$2A - 2B = 0 \Rightarrow B = -\frac{1}{4}$$

So $y_p = -\frac{t^2}{4} - \frac{t}{4} - e^t = -\frac{t}{4}(t+1) - e^t$ and the general solution is:

$$y = y_h + y_p$$

$$y = C_1 + C_2 e^{2t} - \frac{t}{4}(t+1) - e^t$$

Example #2. Solve the differential equation: $y'' - 2y' + y = \dfrac{e^t}{t}$

Not an "easy to guess" function. It is a quotient so the derivatives will get more complicated, making it impossible to list all terms.

1. Homogeneous equation: $y'' - 2y' + y = 0$

   Characteristic equation: $r^2 - 2r + 1 = 0$

   $$(r-1)^2 = 0$$

   $$r = 1, r = 1$$

   $$\Rightarrow y_h = C_1 e^t + C_2 te^t$$

   $y_1 = e^t$ and $y_2 = te^t$

   $y'_1 = e^t$ and $y'_2 = te^t + e^t$

2. Particular solution:

$$W(y_1, y_2)(x) = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \det\begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = e^t\left(te^t + e^t\right) - te^t \cdot e^t = te^{2t} + e^{2t} - te^{2t} = e^{2t}$$

So

$$y_p(x) = -y_1 \int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}dx + y_2 \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}dx =$$

$$= -e^t \int \frac{te^t \cdot \frac{e^t}{t}}{e^{2t}}dt + te^t \int \frac{e^t \cdot \frac{e^t}{t}}{e^{2t}}dt = -e^t \int 1dt + te^t \int \frac{1}{t}dt = -te^t + te^t \ln|t|$$

$y_p(t) = -te^t + te^t \ln|t|$ and the general solution is:

$$\boxed{y = C_1 e^t + C_2 te^t - te^t + te^t \ln|t| = C_1 e^t + C_3 te^t + te^t \ln|t|}$$

You try it:

1. $y'' - y' - 2y = \sin 2x$
2. $y'' - 2y' + y = xe^x$
3. $y'' + y = \sec x$

Solutions:

#1: $y = C_1 e^{-x} + C_2 e^{2x} - \frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$

#2: $y = C_1 e^{x} + C_2 x e^{x} + \frac{1}{6}x^3 e^{x}$

#3: $y = C_1 \cos x + C_2 \sin x + (\cos x)(\ln|\cos x|) + x\sin x$