Comprehensive Summary of Limits and Derivative Calculus

- Limits
- Derivatives
  - Rules, formulas, properties
  - Definition and notation
  - Implicit differentiation
  - Increasing/decreasing
  - Concave up/concave down
  - Extrema
  - Mean value theorem
  - Related rates


**Precise Definition**

We say \( \lim_{x \to a} f(x) = L \) if for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that whenever \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \epsilon \).

**"Working" Definition**

We say \( \lim_{x \to a} f(x) = L \) if we can make \( f(x) \) as close to \( L \) as we want by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) without letting \( x = a \).

**Right hand limit**:

\( \lim_{x \to a^+} f(x) = L \). This has the same definition as the limit except it requires \( x > a \).

**Left hand limit**:

\( \lim_{x \to a^-} f(x) = L \). This has the same definition as the limit except it requires \( x < a \).

**Relationship between the limit and one-sided limits**

\( \lim_{x \to a^-} f(x) = L \Rightarrow \lim_{x \to a} f(x) = L \) \( \lim_{x \to a^+} f(x) = L \Rightarrow \lim_{x \to a} f(x) = L \) \( \lim_{x \to a} f(x) = L \Rightarrow \lim_{x \to a} f(x) \) Does Not Exist

**Properties**

Assume \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) both exist and \( c \) is any number then,

1. \( \lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \)
2. \( \lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) \)
3. \( \lim_{x \to c} [f(x)]^{g(x)} = \left( \lim_{x \to c} f(x) \right)^{\lim_{x \to c} g(x)} \)
4. \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \) provided \( \lim_{x \to c} g(x) \neq 0 \)
5. \( \lim_{x \to c} e^{f(x)} \)
6. \( \lim_{x \to c} a^{f(x)} \)

**Basic Limit Evaluations at \( \pm \infty \)**

Note: \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).

1. \( \lim_{x \to \infty} e^{x} = \infty \) \& \( \lim_{x \to -\infty} e^{x} = 0 \)
2. \( \lim_{x \to \infty} \ln(x) = \infty \) \& \( \lim_{x \to -\infty} \ln(x) = -\infty \)
3. If \( r > 0 \) then \( \lim_{x \to \infty} x^{r} = \infty \)
4. If \( r > 0 \) and \( x \) is real for negative \( x \) then \( \lim_{x \to -\infty} x^{r} = 0 \)
5. \( \lim_{x \to \infty} x^{n} = \infty \) if \( n \) even
6. \( \lim_{x \to \infty} x^{n} = -\infty \) if \( n \) odd
7. \( \lim_{x \to \infty} ax^{r} + bx^{s} + c = \text{sgn}(a)x^{r} \)
8. \( \lim_{x \to \infty} ax^{r} + bx^{s} + c = \text{sgn}(a)x^{r} \)
9. \( \lim_{x \to \infty} e^{x} + cx^{d} = \text{sgn}(a)e^{x} \)

Visit [http://tutorial.math.lamar.edu](http://tutorial.math.lamar.edu) for a complete set of Calculus notes. © 2000 Paul Dawkins
Calculus Cheat Sheet

Derivatives

Definition and Notation

If \( y = f(x) \) then the derivative is defined to be

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]

If \( y = f(x) \) then all of the following are equivalent notations for the derivative.

\[
 f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x) = \frac{dy}{dx} \bigg|_{x=a} = \frac{df}{dx} \bigg|_{x=a}.
\]

Interpretation of the Derivative

If \( y = f(x) \) then,

1. \( m = f'(a) \) is the slope of the tangent line to \( y = f(x) \) at \( x = a \) and the equation of the tangent line at \( x = a \) is given by \( y = f(a) + f'(a)(x-a) \).
2. \( f'(a) \) is the instantaneous rate of change of \( f(x) \) at \( x = a \).
3. If \( f(x) \) is the position of an object at time \( x \) then \( f'(a) \) is the velocity of the object at \( x = a \).

Basic Properties and Formulas

If \( f(x) \) and \( g(x) \) are differentiable functions (the derivative exists), \( c \) and \( n \) are any real numbers,

1. \( (cf)' = cf'(x) \)
2. \( (f \pm g)' = f'(x) \pm g'(x) \)
3. \( (fg)' = f'g + fg' \) - Product Rule
4. \( \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \) - Quotient Rule

Common Derivatives

\[
\begin{align*}
\frac{d}{dx}(x) &= 1 \\
\frac{d}{dx}(c) &= 0 \\
\frac{d}{dx}(x^n) &= nx^{n-1} \\
\frac{d}{dx}(e^x) &= e^x \\
\frac{d}{dx}(\ln x) &= \frac{1}{x} \quad x > 0 \\
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x \\
\frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\tanh x) &= \text{sech}^2 x \\
\frac{d}{dx}(\text{csch} x) &= -\text{coth} x \tan x \\
\frac{d}{dx}(\text{coth} x) &= -\text{csch}^2 x \\
\end{align*}
\]

Derivatives

Basic Properties/Formulas/Rules

\[
\begin{align*}
\frac{d}{dx}(cf(x)) &= cf'(x) \\
\frac{d}{dx}(f(x) \pm g(x)) &= f'(x) \pm g'(x) \\
\frac{d}{dx}(x^n) &= nx^{n-1} \\
\frac{d}{dx}(c) &= 0 \\
(f \cdot g)' &= f'g + fg' \\
\frac{d}{dx}(f'(x)) &= f''(x) \frac{g'(x)}{g(x)} \quad (\text{Quotient Rule})
\end{align*}
\]

Common Derivatives

\[
\begin{align*}
\frac{d}{dx}(x) &= 1 \\
\frac{d}{dx}(c) &= 0 \\
\frac{d}{dx}(x^n) &= nx^{n-1} \\
\frac{d}{dx}(e^x) &= e^x \\
\frac{d}{dx}(\ln x) &= \frac{1}{x} \quad x > 0
\end{align*}
\]

Trig Functions

\[
\begin{align*}
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x \\
\frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\tanh x) &= \text{sech}^2 x \\
\frac{d}{dx}(\text{csch} x) &= -\text{coth} x \tan x \\
\frac{d}{dx}(\text{coth} x) &= -\text{csch}^2 x
\end{align*}
\]

Exponential/Logarithmic Functions

\[
\begin{align*}
\frac{d}{dx}(e^x) &= e^x \\
\frac{d}{dx}(\ln x) &= \frac{1}{x} \quad x > 0 \\
\frac{d}{dx}(\log_a x) &= \frac{1}{x \ln a} \quad x > 0
\end{align*}
\]

Hyperbolic Trig Functions

\[
\begin{align*}
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\tanh x) &= \text{sech}^2 x \\
\frac{d}{dx}(\text{csch} x) &= -\text{coth} x \tan x \\
\frac{d}{dx}(\text{coth} x) &= -\text{csch}^2 x
\end{align*}
\]
The chain rule applied to some specific functions.

1. \( \frac{d}{dx}[f(x)^n] = n[f(x)]^{n-1}f'(x) \)
2. \( \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x) \)
3. \( \frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)} \)
4. \( \frac{d}{dx}\sin[f(x)] = f'(x)\cos[f(x)] \)
5. \( \frac{d}{dx}\cos[f(x)] = -f'(x)\sin[f(x)] \)
6. \( \frac{d}{dx}\tan[f(x)] = f'(x)\sec^2[f(x)] \)
7. \( \frac{d}{dx}\sec[f(x)] = f'(x)\sec[f(x)]\tan[f(x)] \)
8. \( \frac{d}{dx}\tan^{-1}[f(x)] = \frac{f'(x)}{1+[f(x)]^2} \)

Higher Order Derivatives

The second derivative is denoted as:

\[ f''(x) = \frac{d^2f}{dx^2} \]

The nth derivative is denoted as:

\[ f^{(n)}(x) = \frac{d^n f}{dx^n} \]

The derivative of the derivative is called the second derivative.

Implicit Differentiation

Find \( y' \) if \( e^{2x+y} + x^2 y^2 = \sin(xy) + 11x \). Remember, if you have \( y \) as a function of \( x \), then the product/quotient rule and derivatives of \( y \) will use the chain rule. The "trick" is to differentiate normally and every time you differentiate a \( y \) you tack on a \( y' \) (from the chain rule).

After differentiating solve for \( y' \):

\[ e^{2x+y}(2-2y)y' + 2x^2 y y' = \cos(xy) y' + 11 \]
\[ 2e^{2x+y} - 9y e^{2x+y} = 2x^2 y y' + \cos(xy) y' \]
\[ (2e^{2x+y} - 9y e^{2x+y} - 3x^2 y) y' = 11 - 2e^{2x+y} - 3x^2 y \]
\[ y' = \frac{11 - 2e^{2x+y} - 3x^2 y}{2e^{2x+y} - 9y e^{2x+y} - 3x^2 y} \]

Critical Points

\( x = c \) is a critical point of \( f(x) \) provided either:
1. \( f'(c) = 0 \) or 2. \( f'(c) \) doesn't exist.

Increasing/Decreasing

1. If \( f'(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is increasing on the interval \( I \).
2. If \( f'(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is decreasing on the interval \( I \).

Inflection Points

\( x = c \) is an inflection point of \( f(x) \) if the concavity changes at \( x = c \).

Concave Up/Concave Down

1. If \( f''(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave up on the interval \( I \).
2. If \( f''(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave down on the interval \( I \).

Extrema

Absolute Extrema
1. \( x = c \) is an absolute maximum of \( f(x) \) if \( f(c) \geq f(x) \) for all \( x \) in the domain.
2. \( x = c \) is an absolute minimum of \( f(x) \) if \( f(c) \leq f(x) \) for all \( x \) in the domain.

Fermat’s Theorem
If \( f'(x) \) has a relative (or local) extremum at \( x = c \), then \( x = c \) is a critical point of \( f(x) \).

Extreme Value Theorem
If \( f(x) \) is continuous on the closed interval \([a, b]\) then there exist numbers \( c \) and \( d \) so that:
1. \( a \leq c \leq d \leq b \)
2. \( f(c) \) is the abs. max. in \([a, b]\)
3. \( f(d) \) is the abs. min. in \([a, b]\)

Finding Absolute Extrema
To find the absolute extreme of the continuous function \( f(x) \) on the interval \([a, b]\), use the following process:
1. Find all critical points of \( f(x) \) in \([a, b]\).
2. Evaluate \( f(x) \) at all points found in Step 1.
3. Evaluate \( f(a) \) and \( f(b) \).
4. Identify the abs. max. (largest function value) and the abs. min. (smallest function value) from the evaluations in Steps 2 & 3.

Mean Value Theorem
If \( f(x) \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\) then there is a number \( c \) so that:
\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

Newton’s Method
If \( x_n \) is the \( n \)-th guess for the root/zero of \( f(x) = 0 \) then \((n+1)^{th}\) guess is:
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
provided \( f'(x_n) \) exists.
### Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to time using implicit differentiation (i.e., add on a derivative every time you differentiate a function of t). Plug in known quantities and solve for the unknown quantity.

**Ex.** A 15 foot ladder is resting against a wall. The bottom is initially 0 ft away and is being pushed towards the wall at \( \frac{1}{2} \) ft/sec. How fast is the top moving after 12 sec?

![Diagram of a ladder against a wall with notation for x, y, and derivatives]

\( y' \) is negative because \( x \) is decreasing. Using Pythagorean Theorem and differentiating,

\[ x^2 + y^2 = 15^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \]

After 12 sec we have \( x = 10 - 12\left(\frac{1}{2}\right) = 7 \) and
\[ y = \sqrt{15^2 - 7^2} = \sqrt{184} \]. Plug in and solve for \( y' \).

\[ 7\left(-\frac{1}{2}\right) + \sqrt{184} y' = 0 \Rightarrow y' = \frac{7}{4\sqrt{184}} \text{ ft/sec} \]

**Ex.** Two people are 50 ft apart when one starts walking north. The angle \( \theta \) changes at 0.01 rad/min. At what rate is the distance between them changing when \( \theta = 0.5 \) rad?

![Diagram of two people walking with mixed triangles]

We have \( \theta' = 0.01 \) rad/min, and want to find \( x' \). We can use various trig facts but easiest is,

\[ \sec \theta = \frac{x}{50} \Rightarrow \sec \theta \tan \theta \theta' = \frac{x'}{50} \]

We know \( \theta = 0.5 \) rad so plug in \( \theta \) and solve.

\[ \sec(0.5)\tan(0.5)(0.01) = \frac{x'}{50} \]
\[ x' = 0.3112 \text{ ft/sec} \]

Remember to have calculator in radians!

### Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

**Ex.** We’re enclosing a rectangular field with 300 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.

![Diagram of a rectangular field with buildings]

Maximize \( A = xy \) subject to constraint of \( x + 2y = 300 \). Solve constraint for \( x \) and plug into area.

\[ x = 300 - 2y \Rightarrow A = xy(300 - 2y) \]

Differentiate and find critical point(s):

\[ A' = 300y - 4y^2 \]

By 2\textsuperscript{nd} deriv. test this is a rel. max. and so is the answer we’re after. Finally, find \( x \).

\[ x = 300 - 2(125) = 50 \]

The dimensions are then 250 ft \( \times \) 125 ft.

**Ex.** Determine point(s) on \( y = x^3 + 1 \) that are closest to \((0, 2)\).

![Diagram of a cubic function with point (0,2) and line segments to find closest points]

Minimize \( f = d^2 = (x - 0)^2 + (y - 2)^2 \) and the constraint is \( y = x^3 + 1 \). Solve constraint for \( x^3 \) and plug into function.

\[ x^3 + 1 = d^2 = (x - 2)^2 \]
\[ = x^3 - 3x + 3 \]

Differentiate and find critical point(s):

\[ f' = 3x^2 - 3 \Rightarrow x = \frac{1}{2} \]

By the 2\textsuperscript{nd} derivative test this is a rel. min. and so all we need to do is find \( x \) value(s).

\[ x^3 - \frac{1}{2} = 1 - \frac{1}{2} \Rightarrow x = \pm \frac{1}{2} \]

The 2 points are then \( \left(\frac{1}{2}, \frac{3}{2}\right) \) and \( \left(-\frac{1}{2}, \frac{3}{2}\right) \).