I. Definition and Properties of the Unit Circle
   a. Definition: A Unit Circle is the circle with a radius of one \( r = 1 \), centered at the origin \((0,0)\).

   b. Equation: \( x^2 + y^2 = 1 \)

   c. Arc Length
      Since arc length can be found using the formula: \( s = r\theta \)
      (where \( s = \) arc length, \( r = \) radius, \( \theta = \) central angle in radians)
      For the unit circle, since \( r = 1 \), \( s = (1)\theta \)
      Therefore \( s = \theta \)

      The arc length of a sector of a unit circle equals the radian measure of angle \( \theta \).

   d. Circumference: \( C = 2\pi r = 2\pi(1) = 2\pi \)

      The arc length (circumference) of \( 2\pi \) is also the radian measure of the angle corresponding to \( 360^\circ \).
      \( 2\pi \) radians = 360 degrees
      \( \pi \) radians = 180 degrees

   e. Relating Coordinate Values to Trig Functions
      For any point \( P(x, y) \) on the unit circle, 
      \( x = \cos \theta \) and \( y = \sin \theta \) where \( \theta \) is
      any central angle with:
      1) initial side = positive \( x \) axis
      2) terminal side = radius through pt. \( P \)

      In the first quadrant this can be verified:
      \[
      \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} = \frac{y}{1} = y
      \]
      \[
      \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} = \frac{x}{1} = x
      \]
f. The x and y axes can be labeled using the radian measure of the angle $\theta$ which corresponds to the points where the unit circle intersects the axes.

We can also see this graphically:

$y = \sin x$

$y = \cos x$
Finding the sine and cosine values of quadrantal angles is now easy. For example, to find \(\sin \frac{3\pi}{2}\) use the point \((0,-1)\) which corresponds to a central angle of \(\frac{3\pi}{2}\). Since \(\sin \frac{3\pi}{2}\) is the y coordinate of the point, \(\sin \frac{3\pi}{2} = -1\). Similarly, \(\cos \frac{3\pi}{2} = 0\) (the x coordinate of the point).

g. Examples:

i. Find the arc length of a sector in a unit circle with a central angle of 120°.

Solution:
In a unit circle, arc length = central angle measured in radians, \(s = \theta\). Since \(\pi\) radians equals 180°, multiply 120° by the conversion ratio of \(\frac{\pi}{180}\) radians.

\[
120^\circ \left( \frac{\pi}{180} \right) = \frac{2\pi}{3}
\]
Thus the arc length and the measure of \(\theta\) are both \(\frac{2\pi}{3}\) radians.
ii. Find \( \cos \frac{\pi}{2} \) and \( \sin \frac{\pi}{2} \).

**Solution:**
\[ \theta = \frac{\pi}{2} \] implies the angle is a right angle (90°), so \( P = (0,1) \). Hence,
\[ \cos \frac{\pi}{2} = 0 \] (the x coordinate of P) and \( \sin \frac{\pi}{2} = 1 \) (the y coordinate of P).

iii. Find \( \sin(-\pi) \) and \( \cos(-\pi) \)

**Solution:**
If \( \theta = -\pi \), \( P = (-1,0) \). So \( \sin(-\pi) = 0 \) (y value) and \( \cos(-\pi) = -1 \) (x coordinate of P).

**II. More Properties of the Unit Circle**

a. If \( \theta = \frac{\pi}{4} \) (which is equivalent to 45°), then for the point P on the unit circle, \( x = y = \frac{\sqrt{2}}{2} \).

If \( \theta = \frac{\pi}{4} \) (45°), then \( P = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \).

**Explanation:**
\[ x^2 + y^2 = 1 \] (equation of unit circle)
\[ x^2 + x^2 = 1 \] (\( x = y \) since it is an isosceles 45°–45°–90° triangle)
\[ 2x^2 = 1 \]
\[ x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = y \]
b. If $\theta = \frac{\pi}{6}$ (which is equivalent to $30^\circ$), then for the point $P$ on the unit circle, $x = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$.

Using properties of $30^\circ - 60^\circ - 90^\circ$ triangles with hypotenuse of length 1 (since $r = 1$): 

If $\theta = \frac{\pi}{6}$ ($30^\circ$), then $P = \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right)$.

![Diagram of unit circle with point P labeled.] 

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c. If $\theta = \frac{\pi}{3}$ (which is equivalent to $60^\circ$), then for the point $P$ on the unit circle, $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

Using properties of $30^\circ - 60^\circ - 90^\circ$ triangles with hypotenuse of length 1 (since $r = 1$): 

If $\theta = \frac{\pi}{3}$ ($60^\circ$), then $P = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)$.

![Diagram of unit circle with point P labeled.]  

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**NOTE:**  
The larger side, $\frac{\sqrt{3}}{2}$, is always opposite the larger angle, $\frac{\pi}{3}$, and the smaller side, $\frac{1}{2}$, is opposite the smaller angle, $\frac{\pi}{6}$.  

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d. **Examples:**

Find:

(i) \( \sin \frac{\pi}{6} \)

(ii) \( \csc \frac{\pi}{3} \)

(iii) \( \tan \frac{\pi}{4} \)

Solution:

(i) \( \sin \frac{\pi}{6} = \frac{1}{2} \) (y coordinate when \( \theta = \frac{\pi}{6} \))

(ii) \( \csc \frac{\pi}{3} = \frac{1}{\sin \frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \)

(iii) \( \tan \frac{\pi}{4} = \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \)

III. **More on Evaluating Trig Functions Using the Unit Circle**

It is important to recognize the radian measure of the standard angles related to \( \frac{\pi}{6} \), \( \frac{\pi}{4} \), and \( \frac{\pi}{3} \) located in quadrants II, III, and IV.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Reference Angle</th>
<th>QI</th>
<th>QII</th>
<th>QIII</th>
<th>QIV</th>
<th>Point P</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph" /></td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{6} )</td>
<td>( 5\pi )</td>
<td>( 7\pi )</td>
<td>( 11\pi )</td>
<td>( \left( \pm \frac{\sqrt{3}}{2}, \pm \frac{1}{2} \right) )</td>
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<td>( 2\pi )</td>
<td>( 4\pi )</td>
<td>( 5\pi )</td>
<td>( \left( \pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2} \right) )</td>
</tr>
</tbody>
</table>

The signs of the \( x \) and \( y \) values of point P can be determined by knowing the quadrant the angle terminates in. It’s as simple as remembering:
**Examples:**

1. Evaluate $\sin \frac{5\pi}{4}$.

**Solution:**

$\frac{5\pi}{4}$ has a reference angle of $\frac{\pi}{4}$.

Since $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$, it is in quadrant III.

$P = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ (since $x$ and $y$ are negative in QIII)

$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$ (the y coordinate of $P$)

**NOTE:** Our final answer will be negative because only $\tan \theta$ and $\cot \theta$ are positive in the third quadrant.
2. Evaluate \( \cos \frac{5\pi}{6} \).

Solution:
\( \frac{5\pi}{6} \) has a reference angle of \( \frac{\pi}{6} \).

Since \( \frac{5\pi}{6} = \pi - \frac{\pi}{6} \), it is in quadrant II.

\[ P = (-\frac{\sqrt{3}}{2}, \frac{1}{2}) \] (since x negative in QII)

\[ \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \] (the x coordinate of P)

3. Evaluate \( \tan \frac{11\pi}{3} \).

Solution:
\( \frac{11\pi}{3} \) has a reference angle of \( \frac{\pi}{3} \).

Since \( \frac{11\pi}{3} = 2\pi - \frac{\pi}{3} \), it is in quadrant IV.

\[ P = (\frac{1}{2}, -\frac{\sqrt{3}}{2}) \]

\[ \tan \frac{11\pi}{3} = -\frac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \]

IV. Other Facts Derived From the Unit Circle
1. **Fundamental Identity:**
   For any point $P$ on the unit circle, $P = (x, y) = (\cos \theta, \sin \theta)$.
   Substituting $x = \cos \theta$ and $y = \sin \theta$ into the equation of the circle:
   
   \[
   x^2 + y^2 = 1 \\
   (\cos \theta)^2 + (\sin \theta)^2 = 1 \\
   \sin^2 \theta + \cos^2 \theta = 1
   \]

2. **Other identities:**
   The $x$ coordinates of points for $\theta$ and $-\theta$ are the same, so:
   \[
   \cos(-\theta) = \cos(\theta)
   \]
   Therefore $\cos(\theta)$ is an **even** function.

   The $y$ coordinates of points for $\theta$ and $-\theta$ are the opposite, so:
   \[
   \sin(-\theta) = -\sin(\theta)
   \]
   Therefore $\sin(\theta)$ is an **odd** function.

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**Practice Exercises:**
Use the unit circle to answer the following problems.

1. Evaluate:
   a. $\cos \pi$  
   b. $\sin \frac{3\pi}{2}$  
   c. $\tan(-3\pi)$  
   d. $\csc \frac{\pi}{2}$

2. Find the six trigonometric functions values at the following values of $\theta$:
   a. $\frac{5\pi}{3}$  
   b. $\frac{3\pi}{4}$  
   c. $\frac{7\pi}{6}$  
   d. $-\frac{11\pi}{6}$
Solutions: Unit Circle Trig

1. a) -1  
b) -1  
c) 0  
d) 1  

2. 

<table>
<thead>
<tr>
<th>Angle</th>
<th>a. $\frac{5\pi}{3}$</th>
<th>b. $\frac{3\pi}{4}$</th>
<th>c. $\frac{7\pi}{6}$</th>
<th>d. $-\frac{11\pi}{6}$</th>
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</thead>
<tbody>
<tr>
<td>Quadrant</td>
<td>QIV</td>
<td>QII</td>
<td>QIII</td>
<td>QI</td>
</tr>
<tr>
<td>Point P</td>
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<td>$\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>cos $\theta$</td>
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<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>tan $\theta$</td>
<td>$-\sqrt{3}$</td>
<td>$-1$</td>
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<td>$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$</td>
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<tr>
<td>sec $\theta$</td>
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<td>$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$</td>
</tr>
<tr>
<td>cot $\theta$</td>
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<td>$\sqrt{3}$</td>
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