Solving Trigonometric Equations

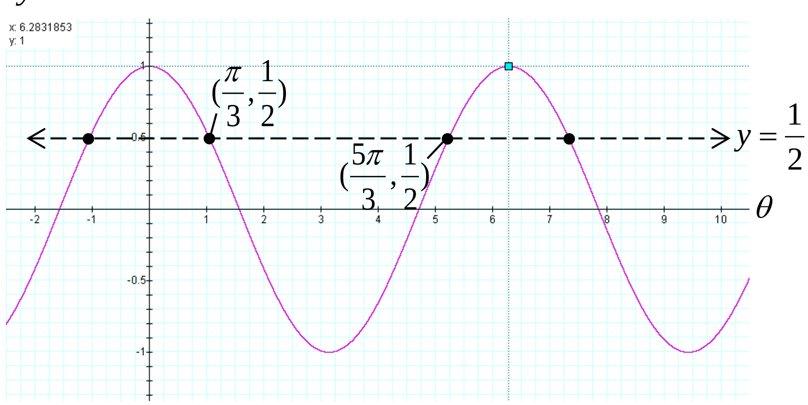
EQUATION SOLVING:

Example 1: Find all possible values of θ so that $\cos\theta = \frac{1}{2}$.

Solution: $\theta = \frac{\pi}{3} + 2\pi n$, $\theta = \frac{5\pi}{3} + 2\pi n$, where n is an integer.

Solution Method #1 – Graphically:
There are an infinite number of solutions which are represented by the θ value of intersection points of the cosine curve and the constant function $y = \frac{1}{2}$.

$y = \cos x$

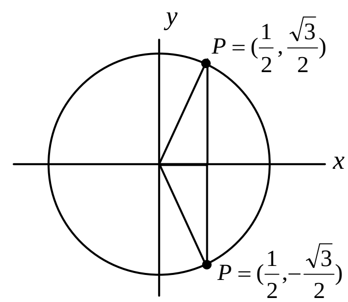


For $0 \le \theta \le 2\pi$, there are two solutions: $\theta = \frac{\pi}{3}$ $(60°)$ and $\theta = \frac{5\pi}{3}$ $(300°)$.

Generalizing, $\theta = \frac{\pi}{3} + 2\pi n$, $\theta = \frac{5\pi}{3} + 2\pi n$, where n is an integer. Thus all solutions differ from the original two solutions by multiples of the period of the cosine function.

Solution Method #2 – Unit Circle Approach:

$\cos\theta = \frac{1}{2}$ occurs when $x = \frac{1}{2}$ for point(s) on the unit circle.

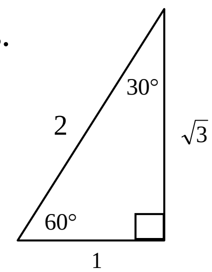


The two points are $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$. The corresponding angles are $\theta = \frac{\pi}{3}$ (QI) and $\theta = \frac{5\pi}{3}$ (QIV).

Generalizing, $\theta = \frac{\pi}{3} + 2\pi n$, $\theta = \frac{5\pi}{3} + 2\pi n$.

Solution Method #3 – Triangle Approach:

$\cos\theta = \frac{1}{2}$ is a special case that involves 30° - 60° - 90° triangles.



Since $\cos\theta = \frac{adj}{hyp} = \frac{1}{2}$, this implies θ must be 60° or $\frac{\pi}{3}$ radians. Generalizing, $\cos\theta$ is also positive in QIV with a reference angle of 60°. Generalizing completely,

$\theta = \frac{\pi}{3} + 2\pi n$, $\theta = \frac{5\pi}{3} + 2\pi n$.

Solution Method #4 – Calculator:

Set the calculator to degree mode. (It will be easier to recognize the answers in degrees, which can then be converted to radian measure.)

Solving $\cos\theta = \frac{1}{2}$ is equivalent to solving:

$$\textbf{inverse } \cos\left(\frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \theta.$$

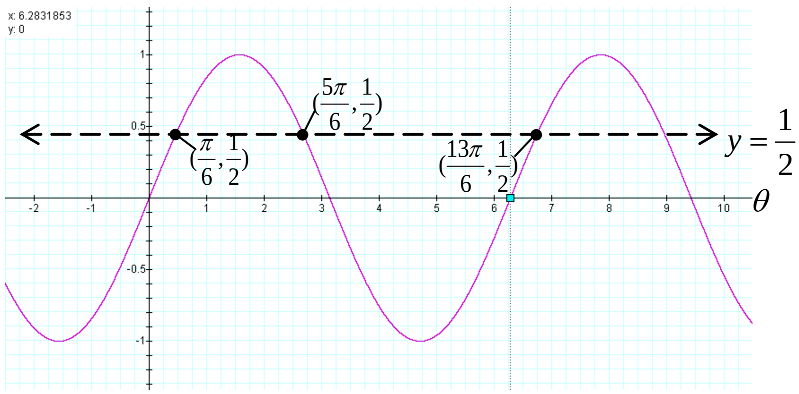
(This is explained in more detail in the handout on inverse trigonometric functions.) Use the **INV** key (or 2nd function key) and the **COS** key with $\frac{1}{2}$ to get an answer of 60°.

Example 2: Find 3 positive and 2 negative solutions for $\sin\theta = \frac{1}{2}$.

Solution: There are many different correct solutions. One solution set is $\theta = \left\{-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}\right\}$

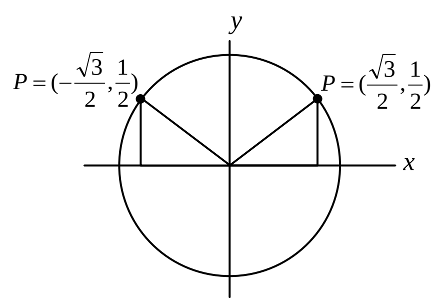
Solution Method #1 – Graphically:
Five solutions are the θ values of the 5 points of intersection of the sine curve and the horizontal line $y = \frac{1}{2}$ shown below.

$y = \sin x$ (3 positive solutions are depicted in the graph)



Thus, $\theta = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$.

Solution Method #2 – Unit Circle Approach:
$\sin\theta = \frac{1}{2}$ occurs when $y = \frac{1}{2}$ for point(s) on the unit circle. The two points are $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ for angles $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. The corresponding angles are $\theta = \frac{\pi}{6}$ (QI) and $\theta = \frac{5\pi}{6}$ (QII).

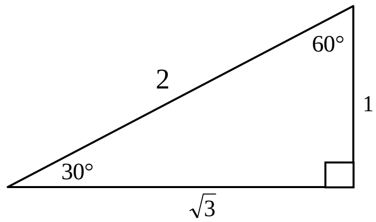


Generalizing, $\theta = \frac{\pi}{6} + 2\pi n$, $\theta = \frac{5\pi}{6} + 2\pi n$.

$\theta = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$

Solution Method #3 – Triangle Approach:

$\sin\theta = \frac{1}{2}$ is a special case that involves 30° - 60° - 90° triangles. Since $\sin\theta = \frac{opp}{hyp} = \frac{1}{2}$, this implies θ must be $30°$ or $\frac{\pi}{6}$ radians. Generalizing, $\sin\theta$ is also positive in QII with a reference angle of $30°$ (so $\theta = \frac{5\pi}{6}$, or $150°$). The other solutions can be found by adding or subtracting multiples of the period.



Solution Method #4 – Calculator:

Set the calculator to degree mode. Solving $\sin\theta = \frac{1}{2}$ is equivalent to solving:

$$\textbf{inverse } \sin\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \theta.$$

(This is explained in more detail in the handout on inverse trigonometric functions.) Use the **INV** key (or 2nd function key) and the **SIN** key with $\frac{1}{2}$ to get an answer of $30°$.

Example 3: Solve for x: $\sqrt{3}\sin x - 2\sin x\cos x = 0,\ 0 \le x < 2\pi$.

Solution: Factor the expression on the left and set each factor to zero.

$$\sin x\sqrt{3} - 2\sin x\cos x = 0$$
$$(\sin x)(\sqrt{3} - 2\cos x) = 0$$

$\sin x = 0$ or $\sqrt{3} - 2\cos x = 0$

$x = 0, \pi$

$\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}$

Answers: $x = 0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}$

Example 4: Solve for $x: \sin^2 x - \sin x - 2 = 0$, $0 \le x < 2\pi$.

Solution: Factor the quadratic expression on the left and set each factor to zero.

$\sin^2 x - \sin x - 2 = 0$

$(\sin x - 1)(\sin x + 2) = 0$

$\sin x - 1 = 0$ or $\sin x + 2 = 0$

$\sin x = 1$ | $\sin x = -2$

$x = \frac{\pi}{2}$ | No solution. (Since the minimum value of sin*x* is -1, it cannot equal -2.)

Answer: $x = \frac{\pi}{2}$

Example 5: Solve for $x: \tan 2x = 1$, $0 \le x < 2\pi$.

Solution: Solving $\tan\theta = 1$ first, we know that $\tan\frac{\pi}{4} = 1$ (QI) and $\tan\frac{5\pi}{4} = 1$ (QIII). So $\theta = \frac{\pi}{4} + \pi n$, where πn is integer multiples of the period of the tangent function.

For our problem:

$\theta = 2x = \frac{\pi}{4} + \pi n$ for $n = \ldots -1, 0, 1, 2, \ldots$

$x = \frac{\pi}{8} + \frac{\pi n}{2}$ (dividing by 2)

$x = \frac{\pi}{8}$ (if $n = 0$), $\frac{5\pi}{8}$ (if $n = 1$), $\frac{9\pi}{8}$ (if $n = 2$), $\frac{13\pi}{8}$ (if $n = 3$)

Note: If $n < 0$ or $n > 3$, the resulting x values are not in the interval of $0 \le x < 2\pi$.

Answer: $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

Problems: Solving Trigonometric Equations

1. Find all possible values of θ so that $\sin\theta = -\frac{1}{2}$.

2. Find one negative and two positive solutions for $\tan x = -1$.

3. Find x, $0 \le x \le 2\pi$, for the following:
   a. $\cos x = \frac{\sqrt{3}}{2}$

   b. $\cos 2x = \frac{\sqrt{3}}{2}$

   c. $2\cos^2 x - \cos x - 1 = 0$

   d. $\cos^2 x - \sin x \cos x = 0$

SOLUTIONS:

1. $\dfrac{7\pi}{6}+2\pi n$, $\dfrac{11\pi}{6}+2\pi n$ for integer n or $210^\circ+360n, 330^\circ+360n$.

2. $\dfrac{-\pi}{4}, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ since the values of tangent are negative in QII and QIV.

3. a) $\dfrac{\pi}{6}, \dfrac{11\pi}{6}$

   b) $2x=\dfrac{\pi}{6}, \dfrac{11\pi}{6}, \dfrac{13\pi}{6}, \dfrac{23\pi}{6}$ so $x=\dfrac{\pi}{12}, \dfrac{11\pi}{12}, \dfrac{13\pi}{12}, \dfrac{23\pi}{12}$

   c) $(2\cos x+1)(\cos x-1)=0$

   $\cos x=-\dfrac{1}{2}$ or $\cos x=1$

   $x=\dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 0, 2\pi$

   d) $\cos x(\cos x-\sin x)=0$

   $\cos x=0$ or $\cos x=\sin x$

   $x=\dfrac{\pi}{2}, \dfrac{3\pi}{2}$ | $x=\dfrac{\pi}{4}, \dfrac{5\pi}{4}$ (note that cos*x* and sin*x* have the same sign in quadrants I and III)